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Mathematics. — *"Theorem on the term by term differentiability of a series."* By Prof. F. SCHUH. (Communicated by Prof. D. J. KORTEWEG).

(Communicated in the meeting of June 28, 1919).

1. *If*

a. the functions $u_1(x)$, $u_2(x)$, ... *are differentiable in the interval* $a \leqq x \leqq b$, *denoted by i,*

b. the series $u'_1(x) + u'_2(x) + \ldots$. *is uniformly convergent in the interval i,*

c. the series $u_1(x) + u_2(x) + \ldots$. *is convergent for a value c of* x *in the interval i,*

d. the functions $u'_1(x)$, $u'_2(x)$, ... *are continuous for the value* $x = p$ *in the interval i,*

then

α. the series $u_1(x) + u_2(x) + \ldots$. *is uniformly convergent in the interval i and*

β. the function $u_1(x) + u_2(x) + \ldots$. *is differentiable for* $x = p$, *its differential coefficient being* $u'_1(p) + u'_2(p) + \ldots$.

Evidently the expression "differentiable" has for $x = a$ or $x = b$ to be taken in the sense of differentiable on the right, resp. on the left.

For this theorem (which is usually deduced in a more restricted form [1]) from the term by term integrability) we shall here give a simple proof resting on the definition of differential coefficient only.

2. Proof of α. If we put $u'_{n+1}(x) + u'_{n+2}(x) + \ldots = R_n(x)$, we have ($x$ belonging to the interval i):

$$u_{n+1}(x) + u_{n+2}(x) + \ldots + u_{n+k}(x) = \{u_{n+1}(c) + u_{n+2}(c) + \ldots + u_{n+k}(c)\} +$$
$$+ \{u'_{n+1}(\xi) + u'_{n+2}(\xi) + \ldots + u'_{n+k}(\xi)\}(x - c) =$$
$$= \{u_{n+1}(c) + u_{n+2}(c) + \ldots + u_{n+k}(c)\} + \{R_n(\xi) - R_{n+k}(\xi)\}(x - c),$$

where $\xi = c + \theta(x - c)$, hence:

$$|u_{n+1}(x) + u_{n+2}(x) + \ldots + u_{n+k}(x)| \leqq$$
$$\leqq |u_{n+1}(c) + u_{n+2}(c) + \ldots + u_{n+k}(c)| + \{|R_n(\xi)| + |R_{n+k}(\quad| \quad (b - a).$$

[1]) i.e. on the assumption, that the functions $u'_1(x)$, $u'_2(x)$, etc. are continuous throughout the interval i.

δ denoting an arbitrary positive number it follows from the assumption c, that a value N_1 can be assigned such that for $n > N_1$ we have $| u_{n+1}(c) + u_{n+2}(c) + \ldots + u_{n+k}(c) | < \frac{1}{3}\delta$. From the assumption b it follows moreover that a number N_2 can be found, such that for $n > N_2$ the inequality $| R_n(\xi) | < \frac{\delta}{3(b-a)}$ holds good.

Let N be the greater of the two numbers N_1 and N_2, then $| u_{n+1}(x) + u_{n+2}(x) + \ldots + u_{n+k}(x) | < \delta$ for every $n > N$ and for every x within the interval i. This establishes α; the assumption d can here be dispensed with.

3. Proof of β. If we put $u_1(x) + u_2(x) + \ldots = \varphi(x)$ and $u'_1(x) + u'_2(x) + \ldots = \psi(x)$ then the assertion β expresses that

$$\lim_{h=0} \frac{\varphi(p+h) - \varphi(p)}{h} = \psi(p). \quad \ldots \quad (1)$$

If we put $u_1(x) + u_2(x) + \ldots + u_n(x) = U_n(x)$ and $u_{n+1}(x) + u_{n+2}(x) + \ldots = \varrho_n(x)$ then

$$\frac{\varphi(p+h) - \varphi(p)}{h} - \psi(p) = \frac{U_n(p+h) - U_n(p)}{h} + \frac{\varrho_n(p+h)}{h} - \frac{\varrho_n(p)}{h} - \psi(p) =$$

$$= U'_n(p + \theta h) + \frac{\varrho_n(p+h)}{h} - \frac{\varrho_n(p)}{h} - \psi(p) =$$

$$= \psi(p + \theta h) - \psi(p) - R_n(p + \theta h) + \frac{\varrho_n(p+h)}{h} - \frac{\varrho_n(p)}{h},$$

hence:

$$\left| \frac{\varphi(p+h) - \varphi(p)}{h} - \psi(p) \right| \leqq \left| \psi(p+\theta h) - \psi(p) \right| + |R_n(p+\theta h)| + \left| \frac{\varrho_n(p+h)}{h} \right| + \left| \frac{\varrho_n(p)}{h} \right|. \quad \ldots \quad (2)$$

In consequence of the assumptions b and d $\psi(x)$ is continuous for $x = p$. Hence, δ denoting an arbitrary positive number, the positive number ε can be assigned such that $| \psi(p+h) - \psi(p) | < \frac{1}{4}\delta$ for $|h| < \varepsilon$; for $| h | < \varepsilon$ we have then also $| \psi(p + \theta h) - \psi(p) | < \frac{1}{4}\delta$.

In consequence of the assumption b we may assign N_1 such that for any $n > N_1$ and any x in the interval i the inequality $| R_n(x) | < \frac{1}{4}\delta$ is satisfied. For $n > N_1$ then $| R_n(p + \theta h) | < \frac{1}{4}\delta$.

Let h be a definite number satisfying $| h | < \varepsilon$ and $h \neq 0$, then the numbers N_2 and N_3 can be so determined that for $n > N_2$ and $n > N_3$ the inequalities $|\varrho_n(p+h)| < \frac{1}{4}\delta |h|$ resp. $|\varrho_n(p)| < \frac{1}{4}\delta |h|$ are satisfied.

Now, if n be chosen larger than the largest of the numbers N_1,

N_2 and N_3, then it follows from (2) that $\left| \frac{\varphi(p+h) - \varphi(p)}{h} - \psi(p) \right| < \delta$ for $|h| < \varepsilon\ (h \neq 0)$, whence (1) may be concluded.

4. Remarks. In the foregoing proof for the differentiability of the function $\varphi(x) = u_1(x) + u_2(x) + \ldots$. for $x = p$ use has been made of the *continuity of the functions* $u'_1(x), u'_2(x)$, *etc. for* $x = p$ *only*; therefore *it was not necessary to suppose their continuity throughout an interval.* Also there was no need to assume the integrability of the functions $u'_1(x), u'_2(x)$, etc., so that the usual proof (where the series $u'_1(x) + u'_2(x) + \ldots$ is integrated term by term) does not apply to the more general formulation as given in N°. 1.

5. When the convergence of the series $u_1(x) + u_2(x) + \ldots\ldots$ throughout the whole interval i is assumed, then it is sufficient, in order to establish β, to assume the semi-uniform convergence (simple-uniform convergence of Dini) of the series $u'_1(x) + u'_2(x) + \ldots\ldots$ in the interval i. This semi-uniform convergence namely is in the first place sufficient to establish the continuity of $\psi(x)$ for $x = p$. The determination of the numbers N_2 and N_3 presents further no difficulty, after which it is possible to attribute to n such a value $> N_2$ and $> N_3$, that the inequality $|R_n(x)| < \frac{1}{4}\delta$ holds for every x in the interval i. It is immaterial whether this inequality also is satisfied for every greater value of n.

In his "Teorica delle funzioni di variabili reali" Dini has demonstrated β by a more complicate transformation of $\frac{\varphi(p+h) - \varphi(p)}{h}$, thereby omitting the assumption d and assuming the series $u_1(x) + u_2(x) + \ldots$. to be convergent throughout the interval i.

6. *In order to demonstrate the differentiability of* $\varphi(x)$ *it was not necessary to assume the absolute convergence of the series* $u'_1(x) + u'_2(x) + \ldots$. By supposing $|u'_n(x)| < c_n$ (c_n independent of x) and the series $c_1 + c_2 + \ldots$. convergent (which includes the uniform absolute convergence of the series $u'_1(x) + u'_2(x) + \ldots$) Porter (Ann. of Math. ser. 2, vol. 3, 1901, p. 19) has proved the differentiability of $\varphi(x)$ for $x = p$ in a very simple way without any supposition concerning the continuity of the functions $u'_n(x)$ *viz.* by making use of the equality

$$\frac{\varphi(p+h) - \varphi(p)}{h} - \psi(p) = \frac{U_n(p+h) - U_n(p)}{h} - U_n'(p) + \\ + \sum_{n+1}^{\infty} u'_j(p + \theta_j h) - \sum_{n+1}^{\infty} u'_j(p).$$

Mathematics. — "*Involutions in a field of circles*". By Prof. JAN DE VRIES.

(Communicated in the meeting of September 27, 1919).

1. In a plane are given three systems of coaxial circles (α), (β), (γ) in each of which the circles are arranged in the pairs α_1, α_2 etc. of an involution. Let δ_1 be the circle which intersects the circles $\alpha_1, \beta_1, \gamma_1$ orthogonally, δ_2 the orthogonal circle of the corresponding circles $\alpha_2, \beta_2, \gamma_2$, then δ_1 and δ_2 are conjugated in an *involutory correspondence* in the field of circles.

Since α_1 coincides twice with α_2, β_1 twice with β_2 and γ_1 twice with γ_2, the involution (δ_1, δ_2) has *eight coincidences*.

In general an arbitrary circle δ_1 is intersected orthogonally by one circle α only. However, when δ_1 belongs to the system (α') of coaxial circles orthogonally intersecting the circles of (α) then α_1, and α_2 also, is an arbitrary circle from (α), whilst β_2 and γ_2 are perfectly defined. In this case every circle δ_2 intersecting β_2 and γ_2 orthogonally corresponds with δ_1.

Hence the *orthogonal systems* (α'), (β'), (γ') of (α), (β), (γ) consist of *singular circles*, i.e. of circles which in the involution are conjugated each to an infinite number of circles.

There is still another way in which δ_1 may be singular. On a circle α the systems (β) and (γ) determine two involutions; since these have one pair in common, on α are to be found the two points of intersection of a circle β with a circle γ. Hence every circle α (or β, or γ) belongs to a triplet $\alpha_2, \beta_2, \gamma_2$, belonging to one system of circles and for which the orthogonal circle accordingly becomes indefinite. The circle δ_1 which intersects the corresponding circles $\alpha_1, \beta_1, \gamma_1$ orthogonally is therefore *singular* and conjugated to every circle of a certain system of coaxial circles.

2. A further investigation of the involution (δ_1, δ_2) becomes comparatively simple, when we make use of a representation of the circles of the field on the points of space, to which Dr. K. W. WALSTRA has attracted attention in 1917 [1]).

In order to obtain this representation we take the plane of our circles as the plane of coordinates $z = 0$. A circle we then represent

[1]) These Proceedings XIX, p. 1130.

by the point on its axis with coordinate z equal to the power of the origin O with respect to the circle.

All circles with radius zero are represented by the points of a paraboloid of revolution **G** (limiting surface) and the images of two orthogonal circles are harmonically separated by **G**. Two reciprocal polar lines are the images of two systems of coaxial circles orthogonal to each other.

The systems $(\alpha), (\beta), (\gamma)$ are represented by three involutions (A_1, A_2), (B_1, B_2), (C_1, C_2) situated on three straight lines a, b, c. The image D_1 of the circle δ_1, which intersects $\alpha_1, \beta_1, \gamma_1$ orthogonally is the pole of the plane $A_1 B_1 C_1$. So we have now to consider an involution (D_1, D_2) of the points of space, which involution is characterized by the property that the polar planes Δ_1 and Δ_2 of D_1 and D_2 meet the given lines a, b, c in the pairs (A_1, A_2), (B_1, B_2), (C_1, C_2) of three given involutions.

3. It is now easy to find the *singular* elements of the involution of circles again. In the first place we observe that A_1 becomes indefinite as soon as Δ_1 passes through a; for Δ_2 now any plane may be chosen which contains the points B_2 and C_2, hence for D_2 any point of the polar line a'_2 of the straight line $a_2 \equiv B_2 C_2$. If Δ_1 is made to revolve about a, then D_1 moves along the polar line a' of a, and a_2 describes a ruled quadric. The line a'_2 also describes a ruled quadric $(a'_2)^2$ of which the polar lines b' and c' of b and c are directrices. It is obvious that to every point of a' a definite straight line of $(a'_2)^2$ is conjugated. Similarly to the *singular lines* b', c' correspond the ruled quadrics $(b'_2)^2$, $(c'_2)^2$.

Secondly D_2 becomes indefinite as soon as A_2, B_2 and C_2 are collinear and therefore situated on a transversal s of a, b, c. When s is made to coincide successively with the generators of the ruled quadric having a, b, c for directrices, then A_1, B_1 and C_1 describe three projective ranges, so that Δ_1 osculates a twisted cubic σ^3, of which the lines a', b' and c' are bisecants. To every point $S \equiv D_1$ of this *singular curve* σ^3 evidently is correlated a line s' *viz.* the polar line of the corresponding line s. The lines s' form a ruled quadric $(s')^2$ with the directrices a', b', c'.

4. If D_1 describes the line l, then Δ_1 revolves about the polar line l', so that A_1, B_1 and C_1 describe projective ranges. A_2, B_2 and C_2 then also describe projective ranges; hence Δ_2 osculates a twisted cubic λ^3, of which a', b' and c' are bisecants. Consequently D_1 and D_2 are conjugated in a *cubic correspondence*.

Since l has two points in common with $(s')^2$, λ^3 rests on σ^3 in two points. The rays of space are in this way transformed into the

fourfold infinity of twisted cubics, which intersect each of the lines a', b', c' and the curve σ^3 twice.

A plane Φ is transformed into a cubic surface passing through a', b', c' and σ^3. The images of two planes have these four lines and the image λ^3 of their line of intersection in common.

5. A tangent plane of the limiting surface $\mathbf{G}$ is the image of the circles which pass through a given point. The involution (δ_1, δ_2) therefore (by § 4) has the following property: *A system of coaxial circles is transformed into a class of circles with index three.*

This class contains *six circles with radius zero* and *three straight lines.* The *singular circles* form *three coaxial systems* (§ 1) and a class with *index* three (§ 3).

To each singular circle a system of coaxial circles is conjugated; these systems form four classes.

The image of a system of coaxial circles contains *eight* singular circles.

6. Evidently the representation of the field of circles on the points of space enables us to deduce from each involution in the latter an involution in the field of circles and vice versa.

A particularly simple involution is obtained as follows. On every ray h which meets OZ at right angles the paraboloid $\mathbf{G}$ determines an involution of conjugated pairs (P,P'). In the field of circles the analogon hereof is the correspondence which conjugates to each other two circles intersecting orthogonally and having the same power with respect to a fixed point O.

The point P', conjugated to P, is the intersection of the ray h with the polar plane π of P. If P lies on OZ, then for h may be taken any perpendicular to OZ passing through P. Since π now is perpendicular to OZ, to P will be conjugated every point of the line of infinity of $z = 0$.

A point of $\mathbf{G}$ lies in its own polar plane and therefore constitutes a *coincidence* of the correspondence. When P reaches the vertex of $\mathbf{G}$ or the point at infinity of OZ, then P' is an arbitrary point of $z = 0$ or of $z = \infty$.

If P moves along a line l, then h describes a ruled quadric ϱ^2 and π a pencil of planes projective with ϱ^2, so the locus of P' is a twisted cubic λ^3. The polar line l' of l meets ϱ^2 in two points P'; each plane through l' contains besides these two points still another point P' not lying on l'. Hence l' is a chord of λ^3. So is l, for its points of intersection with $\mathbf{G}$ are coincidences.

7. To the points P of a plane Ψ correspond the points P' of a cubic surface Ψ^3. Two such surfaces in the first place have the

curve λ^3 in common, which is the image of the line of intersection of the two corresponding planes. In order to obtain a proper insight into the meaning of the figure which they have in common in addition to this, we observe that the involution (P,P') is a particular case of the following correspondence.

Let a quadric surface Φ^2 be given and the pair of polar lines d,d'. Through a point P the straight line t is drawn which meets d and d'; the polar plane π of P defines on t the point P', which we conjugate to P.

The points of intersection of d and Φ^2 we denote by E_1, E_2, those of d' and Φ^2 by E'_1, E'_2. The straight line $E_1 E'_1$ lies in Φ^2; to each of its points P evidently is conjugated any of its points. To each point of d corresponds every point of d'. Thus all the edges of the tetrahedron $E_1 E_2 E'_1 E'_2$ are singular, so that these six lines are conjugated to their points of transit through a plane Ψ. In addition to the curve λ^3 two surfaces Ψ^3 then have these six singular lines in common.

If Φ^2 now again is replaced by **G**, then d becomes the axis OZ, d' is the line at infinity of $z = 0$ and the other four singular lines are to be found in the imaginary lines along which **G** is intersected by $z = 0$ and $z = \infty$.

8. If P is caused to move along a line l, which meets OZ, then h describes a system of parallel lines which is projective to the pencil constituted by the polar line of P with respect to the parabola in the plane through l and OZ. The points P' now are situated on a rectangular hyperbola which by the line at infinity of $z = 0$ is completed to a λ^3.

By the correspondence of the orthogonal circles, which is alluded to in § 6 a *system of coaxial circles* is again transformed into a class *with index three*. The circles with radius zero are coincidences. The two circles of a pair are real only if they have a negative power with respect to O. When O lies without a circle, then the conjugated circle has an imaginary radius.

Physiology. — *"On Polonium Radiation and Recovery of Function."* By Prof. H. ZWAARDEMAKER.

(Communicated in the meeting of October 25, 1919).

Several organs discontinue their functions, when we remove from their environment the potassium-ions, which are always present in the circulating fluids. These functions are restored directly when potassium is replaced by other radio-active atoms in the circulating fluids to a quantity aequi-radioactive to the removed potassium [1]).

It does not matter whether the substitute is an α-rayer or a β-rayer, provided its amount be such that the total radio-activity of the new constituent is about equal to that of the original one. No organ serves our purpose in this experimentation better than the heart of a cold-blooded animal, namely of the frog, because the blood flows on all sides round its cells, which are separated from it only by an endothelium.

There is a rather large number of elements that can replace potassium. Besides rubidium, which was known as such to S. RINGER, my co-workers and I found uranium, thorium, radium, ionium, emanation and actinium (as an admixture to lanthanium and cerium) to be fit substitutes, while of non-radioactive elements only caesium proved serviceable.

However, it is not only the addition of radio-active elements along the blood that can restore the lost function; this can also be effected by radiation from the outside [2]). We succeeded in obtaining this result with mesothorium contained in glass, with radium screened by mica, and unscreened polonium (galvanoplastic on copper). The quantity is of equal order with that which inhibits from the same distance the cultures of bacteria in their growth. 1½ mgr.-hour served for radiation that restored the function; 12 mgr.-hour for excitation of sterility of bacteria.

The recovery of function is, therefore, brought about by radio-activity, anyhow it is in the case of free radiations.

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[1]) Verslag Vol. 25, p. 517 and p. 1096, p. 1282. Vol. 26, p. 555 and p. 776. Proceedings Vol. 19, p. 633 and p. 1043, p. 1161. Vol. 20, p. 768 and p. 773.

[2]) H. ZWAARDEMAKER, C. E. BENJAMINS and T. P. FEENSTRA, Radiumbestraling en hartswerking. Ned. Tijdschr. v. Geneesk. 1916 II, p. 1923 (10 Nov. 1916).

H. ZWAARDEMAKER and G. GRIJNS, Arch. néerland. de physiol., t. 2, p. 500, 1918

It is a moot point as yet whether this action of radiation is direct or indirect.

It may be, namely, that the radiations first liberate the potassium from the potassium-dépôts [1]), which are present in the cardiac muscle and that only then this liberated potassium, diffusing to the circulating fluid, causes the function to revive.

This possibility could not be ignored a priori, it being a fact that during the radiation rather considerable quanta of potassium may quit the blood-cells and perhaps the heart-cells [2]).

One of these days I was in a position to carry out an experimentum crucis.

There is namely antagonism between α- and β-rayers. When applied coincidently with the same activity, they counter-balance each other's action completely.

This antagonism also obtains with external polonium-radiation (α-rayer) and internal appliance of potassium (β-rayers). This became evident when a frog's heart, which had been brought to a standstill by removing the potassium from the circulating fluid, and had recovered its beats again through polonium, ceased beating again after being given a physiological dosis of potassium, whereas it resumed its pulsations both by removal of polonium and by that of potassium.

When the polonium was removed, the potassium gradually regained its influence; when the potassium was removed, only the after-effect of the α-radiation remained.

From the existence of the antagonism polonium-potassium we must conclude that in this case there is a direct action of radiation.

For, if the liberation of potassium-atoms (supposing it to occur)

[1]) In the cells of the cardiac muscle there is a rich store of potassium. It is strange that this permanent substance is of itself not competent to keep up the function. This inactivity cannot be due to incapacity of the radiation of the potassium-dépôt to reach as far as the seat of automaticity. To W. E. RINGER and to myself the radiation seemed to be too penetrating for it. Nothing less than a tissue sheet of 1 m.m. thickness is capable of lessening by half the high penetrating power of potassium. I have therefore been obliged to relinquish my original hypothesis. I am now inclined to look for the explanation in the coincident presence of iron. The cells of the cardiac muscle contain iron atoms where also the potassium-atoms are located. Consequently the miniature magnetic fields surrounding the iron atoms, will dislodge the β-particles of the potassium. It may, therefore, be considered whether perhaps this circumstance constitutes an obstacle for outward radiation.

Biologically, various explanations are given, starting from the inactivity of continuous causes and the stimulation of temporary ones.

[2]) Researches not published yet. They will be recorded elsewhere.

should have had to serve as an intermediary, it would be impossible to conceive that the addition of a small quantity of potassium, entirely within physiological limits, should have doomed the polonium-heart to a standstill. On the other hand, if the supposition had come true, the liberated and the newly added potassium-atoms would have aided each other and would have maintained the function, instead of disturbing it as was the case now in consequence of the joint action of polonium-radiation and the internal circulating potassium.

Physiology. — "*Researches on scopolamin-morphin narcosis*". By Dr. W. STORM VAN LEEUWEN and Miss M. v. d. MADE. (Communicated by Prof. MAGNUS).

(Communicated in the meeting of September 27, 1919).

Much work has been done on scopolamin-morphin narcosis ever since SCHNEIDERLIN [1]) introduced it into medical practice (1900). One of the principal questions that occupied the workers in this field was, whether the administration of a mixture of these two poisons brings about a "potentiated synergism", which BÜRGI defines as being a stronger effect of a mixture of poisons than the action the component parts alone could lead us to expect. In order to ascertain whether in the scopolamin-morphin mixture "potentiation" occurs, HAUCKOLD [2]) has performed many experiments with rabbits, KOCHMANN [3]) with dogs and SCHNEIDERLIN with men.

HAUCKOLD administered subcutaneously to rabbits respectively morphin, scopolamin, and morphin-scopolamin and detected "potentiation". He recorded that 5 mgr. of morphin + 0,5 mgr. of scopolamin per kg. can produce narcosis in a rabbit, while 10 mgr. of morphin and 200 mgr. of scopolamin injected separately did not produce a narcotic effect. From this HAUCKOLD concluded that, though scopolamin *per se* does not bring about narcosis in a rabbit, it is nevertheless capable of activating a non-narcotic morphin-dosis. This assertion, however, appeared to be based on erroneous observation, first because we have demonstrated by a method to be discussed lower down, that 0,5 mgr. of scopolamin as well as 5 mgr. of morphin produce decidedly a narcotic effect and secondly, because also with HAUCKOLD's method, we found no potentiation, but merely simple addition of effects.

HAUCKOLD administered the scopolamin, the morphin, and the mixture scopolamin-morphin to rabbits subcutaneously and then

[1]) SCHNEIDERLIN. Eine neue Narkose. Ärztl. Mitt. aus u. für Baden. Mai 1900, quoted from HAUCKOLD SCHNEIDERLIN. Die Skopolamin-Morphin-Narkose. Münch. Med. Wochenschr. 1903. N°. 9, pag. 371.

[2]) E. HAUCKOLD. Ueber die Beeinflussung von Narkoticis durch Skopolamin. Zeitschr. f. exp. Path. u. Ther. Bd. 7, pag. 743, 1910.

[3]) M. KOCHMANN. Ueber die therapeutischen Indikationen des Skopolaminum hydrobromicum. Die Therapie der Gegenwart. 1903, pag. 202.

ascertained whether or no a narcosis ensued. We have repeated these experiments, but since — as also HAUCKOLD observes — the depth of the narcosis is difficult to judge in rabbits, we altered the technique by administering on the same day the various poisons to a series of about twenty rabbits, almost simultaneously. Every quarter of an hour the condition of the animals was observed and noted down, the observer not knowing what poison had been injected into the animal under observation.

In this way we made the following experiments:

6 rabbits were given 10 mgr. of morphin per kg. subcutaneously.

6 rabbits were given 1 mgr. of scopolamin per kg. subcutaneously.

6 rabbits were given 5 mgr. morphin + 0,5 mgr. scopolamin per kg. subcutaneously.

All the animals were examined regularly during $2^1/_2$—3 hours.

It thereby appeared that 1 mgr. of scopolamin had only a slight narcotic effect. The action of 10 mgr. of morphin was manifest; that of 0,5 mgr. of scopolamin + 5 mgr. of morphin was less marked than that of 10 mgr. of morphin alone; consequently "potentiation" was out of the question.

After this negative result we examined the narcotic effect of scopolamin and morphin also by another method. A so-called isolated rectusfemoris preparation was made on decerebrated rabbits and the influence was recorded of morphin, of scopolamin, and of morphin + scopolamin on the homolateral contraction-reflex of the rabbit. After a slight technical correction this method, which had already often been applied to cats [1]), appeared to be well-adapted for rabbits.

The reflexes elicited in decerebrated rabbits by the faradic stimulus were registered on a Kymograph; and afterwards the results of every experiment were plotted. An instance of the influence of 5 mgr. of morphin on the homolateral contraction-reflex of the rabbit is given in Fig. 1.

In these experiments series of five rabbits were given 0,5 mgr. of scopolamin, or 10 mgr. of morphin or 0,5 mgr. of scopolamin + 5 mgr. of morphin and in all these cases the injections were not given till it appeared that the reflexes elicited by equi-intense stimulation were of the same magnitude. In case scopolamin + morphin was given, first the scopolamin was injected and 20 minutes later the morphin.

The effect of the injections on the magnitude of the reflexes were

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[1]) W. STORM VAN LEEUWEN, Quantitative pharmakologische Untersuchungen über die Reflexfunktionen des Rückenmarks bei Warmblütern. I. Mitt Pflügers Arch. Bd. 154, page 307. 1913. III. Mitt. Pflügers Arch. Bd. 165, p. 84. 1916.

noted for the scopolamin after 40 min., for the mixture scopolamin + morphin 40 min. after the scopolamin-injection, and for the morphin after 20 minutes.

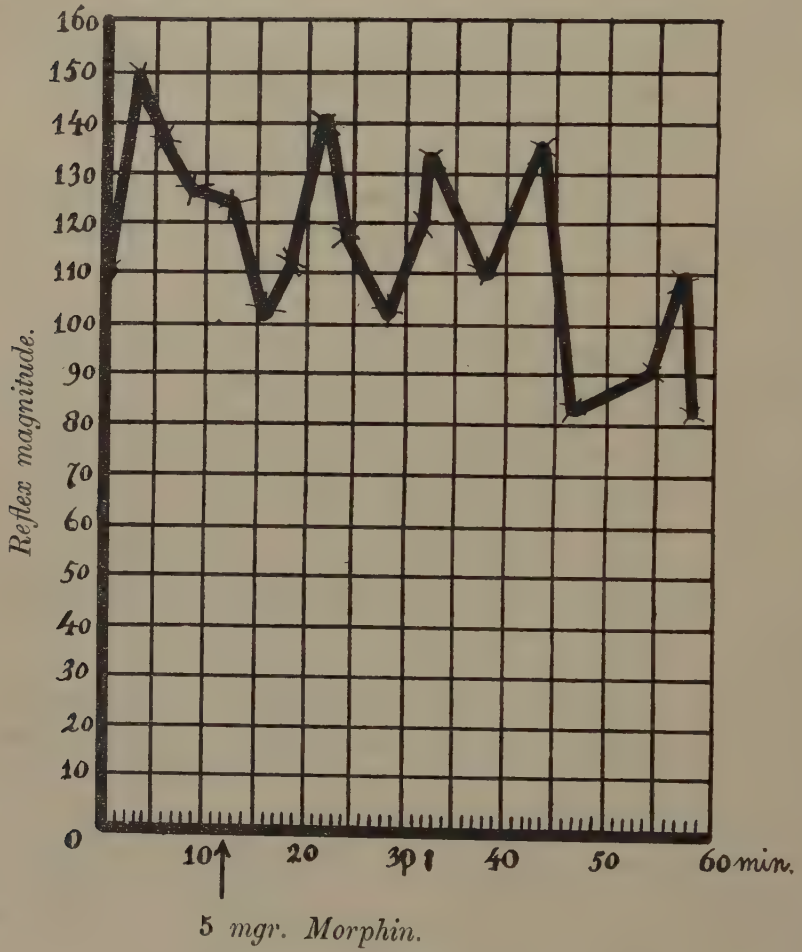

Fig. 1.

The results are as follows:
0,5 mgr. of scopolamin causes after 40 min. a decrease of 37 %
5 mgr. of morphin „ „ 20 „ „ „ „ 15 %
0,5 mgr. of scop. + 5 mgr. of morphin causes after 40 minutes a decrease of 32 %.

These percentages require correction, as it appeared from a number of control-experiments that the magnitude of the reflexes diminishes spontaneously if no poison is administered, viz. 9 % after 40 minutes and 4 % after 20 minutes, so that the above values are respectively 28 %, 11 % and 23 %.

It is evident therefore, that also if this procedure is followed there is no trace of "potentiation", the value obtained in the mixture-experiments not being higher, but lower than the sum of the actions of morphin and scopolamin separately.

HAUCKOLD and many others assume that small doses of scopolamin do not anaesthetize a rabbit. Our experiments go to show that 0.5 mgr. of scopolamin has a distinct narcotic effect upon the magnitude of the reflex. In conjuction with Dr. G. LILJESTRAND we ascertained the influence of various doses of scopolamin on a spinal reflex of the decerebrated rabbit. The result was that this narcotic effect does not increase continually with an increase of the dosis, but soon reaches an optimum and even decreases again after this (see fig. 2 firm line). It seems that with the higher doses a stimulating effect is added to the narcotic effect.

It might be generally assumed that a twice larger dosis yields a twice stronger effect. This, however, is not the case with many of the alkaloids. When plotting the relation between the dosis per kg. of the animal and the effect of such a poison, the doses along the abscissae and the (narcotic) effect along the ordinates, a curve is produced, which first ascends abruptly, and then proceeds nearly horizontally. This is seen distinctly in fig. 2 (dotted line), borrowed from a paper by LILJESTRAND, v. D. MADE and STORM VAN LEEUWEN [1]), in which the full line illustrates the narcotic effect of scopolamin in various doses (concentration-effect curve of scopolamin).

Besides HAUCKOLD, KOCHMANN also studied this problem. He experimented with dogs and believed that he had detected "potentiation". Because we did not succeed in finding "potentiation" either with HAUCKOLD's method, nor with the one commonly used at our institute (influence on reflexes of decerebrated animals), we have also put to the test KOCHMANN's experiments.

For this purpose series of from 3 to 6 dogs were given subcutaneously morphin, or scopolamin or morph. + scop. The doses were calculated per kg. animal.

An initial experiment was performed with the same dosis KOCHMANN had used. KOCHMANN asserts that 5 mgr. of morphin and 0.5 mgr of scopolamin (we suppose his doses to be given per kg. animal, though the writer does not say so) do not of themselves produce any narcotic effect on the dog, but that their joint action puts the animal under a profound narcosis. It must be argued that this occurred only in few cases. Our initial results seemed to substantiate KOCHMANN's findings, for it appeared that in a dog 5 mgr of morphin + 0.5 mgr

[1]) LILJESTRAND, v. D. MADE and STORM VAN LEEUWEN. Zur Konzentrations-Wirkungskurve des Skopolamins. Appears in Pflügers Arch. 1919.

of scopolamin exerted as strong an action as 10 mgr of morphin by itself, while 0.5 mgr of scopolamin exerts a stimulating rather

Paralysing effect of Scopolamin on the homolateral contraction-reflex in the decerebrated rabbit.

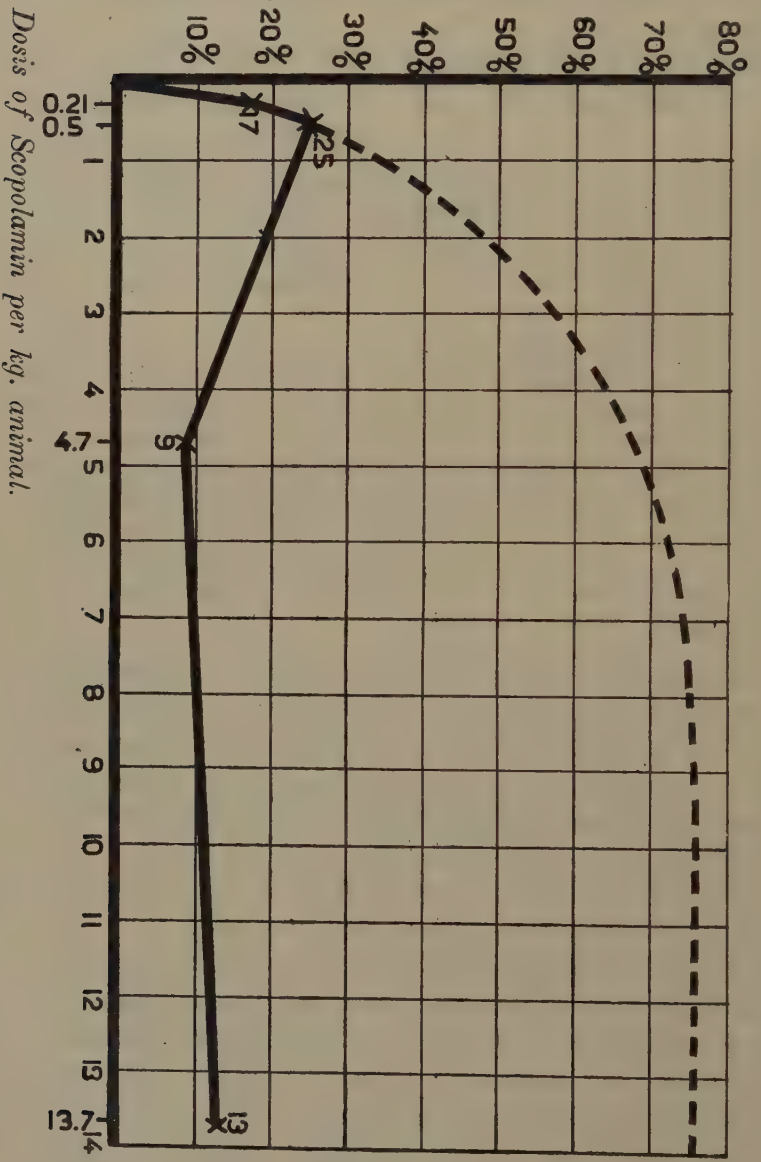

Fig. 2.

than a paralysing influence. On closer investigation it appeared, however, that the case is different, for when 5 mgr. of morphin was given alone (i.e. without scopolamin) its effect proved not to be different from that of 10 mgr. of morphin. This finding induced us

to investigate the action of different quanta of morphin on the dog. The results obtained were that doses smaller than 1,5 mgr. had but little narcotic effect or none at all; that 1,5 mgr. yields a distinct action, 2,5 mgr. a narcotic action, which is considerably stronger, while the effect of 5 mgr. differs little from that of 2,5 and finally that, as said above, the influence exerted by 10 mgr. is about equal to that of 5 mgr. If we should draw a curve of this peculiar relation between action and concentration, a so-called concentration-effect curve, it would again rise abruptly at the outset and then again proceed about horizontally, just like the curve representing the action of morphin and of scopolin on the reflexes of the rabbit.

After it had thus appeared that the action of 5 mgr. of morphin + 0,5 mgr. of scop. agrees with that of 10 mgr. of morphin, but still is not greater than that of 5 mgr. of morphin alone, we tried to find out whether the action of smaller doses of morphin was intensified by scopolamin. An injection of 2,5 mgr. of morph. per kg. was given to two dogs I and II, and to two other dogs III and IV 2,5 mgr. of morph. + 0,5 mgr. of scop. per kg. was administered. The narcosis of dog III was as profound as that of I and II, but that of IV was considerably *less* profound than that of I and II. The inevitable conclusion, therefore, is that scopolamin inhibited the action of morphin, so it did not bring about "potentiation" at all. This result was confirmed by a series of experiments with dogs, in which some received 1,5 mgr. of morphin and others 1,5 mgr. of morph. + 0,5 mgr. of scop. The narcosis with the former group was invariably more profound than that of the second.

It follows from these experiments that although narcotic symptoms are generated by scopolamin in doses of from 0,5 to 1 mgr. per kg., this poison has also a distinct stimulating effect on the dog. When small doses of morphin are mixed with scopolamin, the result is the algebraic sum of the effects of the two components and the narcotic effect of the morphin antagonizes in part the stimulating effect of the scopolamin. With larger doses of morphin the stimulating effect of the scopolamin falls back, while it would seem that the narcotic effect of morphin is sometimes reinforced in a small degree by the scopolamin, however in such a small degree that it cannot be called "potentiation", but is merely to be considered as a simple addition.

Upon the evidence founded upon very accurate experiments SCHNEIDERLIN concluded that in men the scop-morph. combination produces a real "potentiated" narcotic effect. He administered a rather

large dosis of morphin to some patients, after a few days a dosis of scopolamin, and again some days later half the dosis of each poison. In control-experiments, some patients were first given the morph. + scopol. and afterwards the two poisons separately. In all cases the administration of the mixture resulted in a general narcosis, which did not occur after morphin or scopolamin alone. This then is a case of true "potentiation".

This finding does, however, not yield great profit for the clinic, since this result is not always obtained and, as to sensitiveness to scopolamin, patients differ too much to render a correct dosage possible.

Summary.

Morphin produces a narcotic effect in rabbits, in dogs (and also in men), but the curve representing the relation between dosis and effect is a parabola, which means that with the smaller doses a small increase in the doses causes a considerable increase in the narcotic effect, whereas with the larger doses a similar increase of dosis brings about a much smaller increase of the narcotic action.

Scopolamin narcotizes the rabbit. The concentration-effect curve concurs with that of morphin. Its stimulating effect is obvious in the dog, in man its action is evidently narcotic.

No "potentiation" is obtained in the rabbit with the scopolamin-morphin mixture, either by our reflex-method or by HAUCKOLD's method. Neither was "potentiation" dedected in the dog.

According to SCHNEIDERLIN "potentiation" occurs in man, but it is presumably not constant.

Physiology. — *"On the question whether or no Darkness-nystagmus in dogs originates in the Labyrinth"*. By A. DE KLEYN M.D. and C. R. J. VERSTEEGH M.D. (Communicated by Prof. R. MAGNUS).

(Communicated in the meeting of September 27 1919).

An extensive clinical examination of miners induced OHM to believe that the nystagmus, common among this class of people, originates in the labyrinth.

RAUDNITZ's discovery that nystagmus is elicited in dogs after a prolonged sojourn in the dark has enabled OHM to adduce experimental evidence for the above hypothesis.

He starts from the consideration "dass der bei jungen Hunden durch Dunkelheit hervorgerufene Nystagmus in Bezug auf Ablauf, Ausschlag und Dauer der Zuckung, den Einflusz der Ruhe und Bewegung mit dem Augenzittern der Bergleute volkommen übereinstimmt".

He then proceeds by describing some experiments in which he has tried to perform bilateral labyrinth-extirpation in dogs suffering from darkness-nystagmus.

In this effort he failed, except once, when the laboratory animal did not show any more symptoms of nystagmus. However, it was so weak that it died some days afterwards.

From this doubtful success OHM concludes that the labyrinthal origin of darkness-nystagmus has been established.

By a more effectual method of labyrinth-extirpation we were so fortunate as to demonstrate that:

Darkness-nystagmus in young dogs (RAUDNITZ, OHM) *is not of labyrinthal origin.*

We found that:

1. the existing darkness-nystagmus persisted after bilateral labyrinth-extirpation.

2. darkness-nystagmus can be elicited even after bilateral labyrinth-extirpation.

Fig. 1 shows the curve of this nystagmus. We see that the typical darkness-nystagmus is interrupted by a few spontaneous greater movements. The registration was performed by attaching a wire

through the anaesthetized cornea and by thus transmitting the movements to a lever.

3. the darkness-nystagmus behaves towards a super-added vestibulary nystagmus in a different way from a vestibulary nystagmus, i.e. if we impart a vestibulary stimulus to a dog suffering from darkness-nystagmus, thus evolving an additional vestibulary nystagmus, the two forms of nyst. will be seen to persist concurrently, without exerting any influence upon each other.

If, however, we superadd to an existing vestibulary nyst. (e.g. caloric) a second vestibulary nyst. (e.g. rotatory nyst.) the first nyst.-deflections will be seen to be slightly irregular and of different magnitude, but the effect would seem to be rather a resultant of two movements.

Fig 1.

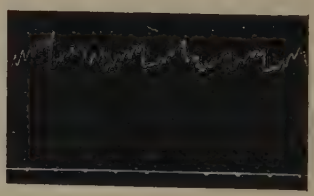

Fig 2.

Fig. 2 illustrates the typical instance of what is called the combination of darkness-nystagmus and caloric nyst. The minor deflections represent the movements of the quietly proceeding darkness-nystagmus; the larger waves show the eye-movements provoked by irrigating the ear with cold water.

We cannot say here in how far OHM is justified in looking for the origin of the miners' nystagmus in the labyrinth, but we feel confident in asserting that he is wrong in assigning the cause of the darkness-nystagmus in dogs to the labyrinth.

(*Pharmacological Institute of the Utrecht-University*).

Mathematics. — "*Bestimmung der Klassenzahl der Ideale aller Unterkörper des Kreiskörpers der m-ten Einheitswurzeln, wo die Zahl m durch mehr als eine Primzahl teilbar ist*". (Zweiter Teil. [1])). By Dr. N. G. W. H. BEEGER. (Communicated by Prof. W. KAPTEYN).

(Communicated in the meeting of October 25, 1919).

Beweis: Wir müszen zwei Behauptungen des Gliedes rechter Hand beweisen: 1°. dasz der Wert des darin auftretenden Symbols eine $\frac{f_1}{d'_1}$-te Einheitswurzel ist; 2°. dasz, im Producte, jede solche Einheitswurzel $\frac{e_1 d_1 d'_1}{r}$ Mal vorkommt. Denn wenn dies bewiesen ist, ergibt sich daraus dasz das Glied rechter Hand gleich

$$\left(1 - \frac{1}{l_1^{\frac{f_1}{d'_1} s}}\right)^{\frac{e_1 d_1 d'_1}{r}}$$

ist, und dieser Ausdruck ist, wegen Satz 7, dem Gliede linker Hand der zu beweisenden Gleichheit, gleich. Die Zahlen e_1, f_1 u.s.w. haben hier dieselbe Bedeutung wie im Satz 7.

Der Beweis der ersten Behauptung folgt aus demjenigen welches im Satz 9 bewiesen ist.

Um den Beweis der zweiten Behauptung zu erbringen, nehmen wir an, dasz

$$\left[\overbrace{b_{01}, b_{*1}, 0, b_{21} \ldots}^{l_1}\right] = \left[\overbrace{b_{02}, b_{*2}, 0, b_{22} \ldots}^{l_1}\right],$$

Nach Einführung der Werte der Symbole findet man, da die Differenzen der übereinstimmenden Zahlen zweier Systeme der b wiederum ein System der b bilden, dasz dieses System der b welches wir andeuten durch $b_0, b_*, 0, b_2, \ldots\ldots$, den Congruenzen:

$$\left.\begin{aligned} &\tfrac{1}{2}\varphi\left(\frac{m}{l_1^{h_1}}\right) a_0 b_0 + 2\varphi\left(\frac{m}{2^{h_*} l_1^{h_1}}\right) a_* b_* + \varphi\left(\frac{m}{l_2^{h_2} l_1^{h_1}}\right) a_2 b_2 + \ldots \\ &\qquad \equiv 0 \bmod \varphi\left(\frac{m}{l_1^{h_1}}\right) \end{aligned}\right\} \quad (8)$$

[1]) Fortsetzung von "Proceedings" Vol. 22. S. 331.

genüge leistet, wo $a_0, a_*, \ldots$ die Exponente bedeuten, welche in § 4 bestimmt sind. Wir müszen nun die Anzahl der Systeme der b berechnen, welche diesen Congruenzen (8) genüge leisten. Dazu bestimmen wir zunächst wieviel der gegebenen Systeme der b aus § 1, (1) eine Zahl $b_{1n} = 0$ enthalten. Alle Zahlen b_{1n} sind teilbar durch ihren grösszten gemeinsamen Teiler d_1. Es gibt also $\frac{\varphi_1}{d_1}$ verschiedene Zahlen b_{1n}, und von diesen ist nur eine $= 0$. Weil es im ganzen $\frac{\varphi}{r}$ Systeme gibt, kommen darunter $\frac{\varphi}{r} : \frac{\varphi_1}{d_1}$ Systeme vor, welche $b_{1n} = 0$ haben.

Alle Systeme der Zahlen a, welche die Gruppe g bilden, leisten allen Congruenzen (3) genüge, wenn man darin $b_{1n} = 0$ setzt. Den Modulus dieser Congruenzen kann man dabei reduciren zu $\frac{\varphi}{\varphi_1}$. Nach der Bemerkung am Ende des Kapitels II § 3, ist die Anzahl der verschiedenen Systeme der a, wenn man a_1 ausser Acht lässt, also gleich

$$\frac{\varphi}{\varphi_1} : \left(\frac{\varphi}{r} : \frac{\varphi_1}{d_1}\right) = \frac{r}{d_1}.$$

Weiter bemerken wir, dasz die Systeme der b welche (8) genügen, auch den $\frac{r}{d_1}$ Congruenzen genügen, welche man aus (8) ableiten kann, indem man darin $a_0, a_*, \ldots$ durch die, eben berechneten, Systeme der a ersetzt. Auszerdem genügen die gesuchten Systeme der b noch den $\frac{f_1}{d'_1}$ Congruenzen die man aus (8) erhält indem man die darin auftretenden a nacheinander durch ihren 2-, 3-, ... $\frac{f_1}{d'_1}$-fachen Wert ersetzt. Denn wegen Kap. II § 4 gehören diese neuen Systeme der a nicht zu den erstgefundenen, weil die $\frac{f_1}{d'_1}$ ersten Potenzen der Zahl $l_1 + n\frac{m}{l_1^{h_1}}$ nicht zu g gehören.

Wenn man nun die Gruppe der zuerstgefundenen Systeme der a mit der Gruppe der zuletztgefundenen Systeme multipliziert, so bekommt man eine Gruppe von $\frac{rf_1}{d_1 d'_1}$ Systeme der a aus welchen sich ebensoviele Congruenzen (8) ergeben, welche die gesuchten Systeme der b genüge leisten. Wegen der Bemerkung am Ende des Kap. II § 3 gibt es also

$$\varphi\left(\frac{m}{l_1 h_1}\right) : \frac{r f_1}{d_1 d'_1} = \frac{e_1 d_1 d'_1}{r}$$

gesuchte Systeme der b.

Hiermit ist nun gezeigt worden dasz jede $\frac{f_1}{d'_1}$-te Einheitswurzel welche in dem Gliede rechter Hand der, im Satze genannten, Gleichheit auftritt, darin auch $\frac{e_1 d_1 d'_1}{r}$ Mal auftritt.

Nun ist aber die Anzahl der Factoren des Gliedes rechter Hand, wie schon berechnet ist, gleich $\frac{\varphi}{r} : \frac{\varphi_1}{d_1}$; und da $\frac{f_1}{d_1} \cdot \frac{e_1 d_1 d'_1}{r} = \frac{\varphi}{r} : \frac{\varphi}{d_1}$ ist, so wird auch *eine jede* $\frac{f_1}{d'_1}$-te Einheitswurzel $\frac{e_1 d_1 d'_1}{r}$ Mal im Produkte auftreten, was zu beweisen war.

Satz 11. Wenn m gerade ist, so ist

$$\prod_{\mathfrak{l}}\left(1 - \frac{1}{n(\mathfrak{l})^s}\right) = \prod_{n} \left\{1 - \left[\frac{2}{b_{1n}, b_{2n}, \ldots}\right] \frac{1}{2^s}\right\}$$

Das erste Produkt erstreckt sich über alle Primideale $\mathfrak{l}$ welche in 2 aufgehen im Körper k. Das zweite Produkt erstreckt sich über alle diejenigen Systeme der b in welchen $b_{0n} = b_{*n} = 0$ ist.

Der Beweis ist ganz in Übereinstimmung mit den beiden Vorigen. Zuerst beweist man dasz das Symbol eine $\frac{f_*}{d'_*}$-te Einheitswurzel ist. Weiter ist es notwendig drei Fälle zu unterscheiden:

1°. $h_* \geqslant 3$ und $b_{0n} + b_{*n}$ nicht für alle Werte von n gerade.

2°. $h_* \geqslant 3$ und $b_{0n} + b_{*n}$ wohl für alle Werte von n gerade.

3°. $h_* = 2$.

IV. *Berechnung der Klassenzahl der Ideale des Unterkörpers k.*

§ 8. *Hülfssatz und Ableitung der vorläufigen Formel.*

1. Es ist $\sum_{n=1}^{m} \left[\frac{n}{\sim}\right] = 0$[1].

Beweis: Es sei a prim zu m; dann ist $\left[\frac{a}{\sim}\right] \neq 0$ und $\neq 1$; daher ist.

$$\left[\frac{a}{\sim}\right] \sum_{n=1}^{m} \left[\frac{n}{\sim}\right] = \sum_{u=1}^{m} \left[\frac{n\,a}{\sim}\right].$$

Es durchläuft $n\,a$ zugleich mit n ein vollständiges Restsystem

[1]) Zur Abkürzung lasse ich die Buchstaben b im Symbole weg.

$(mod\ m)$. Die letzte Summe ist daher gleich $\sum_{n=1}^{m} \left[\frac{n}{m}\right]$. Hieraus ergibt sich leicht der Beweis.

Für die Klassenzahl H gebrauchen wir den bekannten Ausdruck[1]):

$$H = \frac{1}{\varkappa} \lim_{s=1} (s-1) \prod_{\mathfrak{p}} \frac{1}{1-n(\mathfrak{p})^{-s}}$$

wo $\mathfrak{p}$ alle Primideale des Unterkörpers k durchläuft. Wenn wir nun die Sätze 9, 10 und 11 benützen und die Factoren, welche sich beziehen auf dem System $b_{0n} = b_{*n} = b_{1n} = \ldots = 0$, von den anderen Factoren abscheiden, so findet sich:

$$H = \frac{1}{\varkappa} \lim_{s=1} (s-1) \prod_{n=2}^{\varphi/r} \prod_{p} \frac{1}{1-p^{-s}} \prod_{p} \left\{1 - \left[\frac{p}{b_{0n}, b_{*n}, b_{1n} \ldots}\right] \frac{1}{p^s}\right\}^{-1}.$$

Es ist hierbei angenommen dasz $b_{01} = b_{*1} = b_{11} = b_{21} = \ldots = 0$ ist. Wir wissen nunmehr dasz:

$$\lim_{s=1} (s-1) \prod_{p} \frac{1}{1-p^{-s}} = 1.$$

Weiter entwickeln wir jeden Faktor des dritten Produktes auf bekannte Weise in einer DIRICHLET'schen Reihe und multiplizieren all' diese Reihen. Das Ergebnisz ist:

$$H = \frac{1}{\varkappa} \lim_{s=1} \prod_{i=2}^{\varphi/r} \sum_{n=1}^{\infty} \left[\frac{n}{b_{0i}, b_{*i}, b_{1i}, \ldots}\right] \frac{1}{n^s}$$

Hierin setzt man

$$\frac{1}{n^s} = \frac{1}{\Gamma(s)} \int_0^{\infty} e^{-nx} x^{s-1}\, dx$$

und

$$\sum_{t=1}^{m} \left[\frac{t}{m}\right] x^t = F(x)$$

und wenn man noch Gebrauch macht von der Gleichheit $\left[\frac{n}{m}\right] = \left[\frac{n'}{m}\right]$ wenn $n \equiv n' (mod\ m)$:
so findet man

$$H = \frac{1}{\varkappa} \prod_{n=2}^{\varphi/r} \int_0^1 \frac{F(x)}{x(1-x^m)}\, dx$$

wenn man auszerdem den ersten Hülfssatz dieser § benutzt. Nach Zerlegung in Partialbrüchen kann man die Integration durchführen und findet dann:

---

[1]) „H". Satz 55 und § 27.

$$H=\frac{1}{\varkappa}\prod_{n=2}^{\varphi/r}-\frac{1}{m}\sum_{k=1}^{m}F\left(e^{\frac{2k\pi i}{m}}\right)\left\{log\frac{e^{\frac{k\pi i}{m}}-e^{\frac{-k\pi i}{m}}}{i}+\tfrac{1}{2}\pi i-\frac{k\pi i}{m}\right\}.$$

Nun ist noch

$$\sum_{k=1}^{m}F\left(e^{\frac{2k\pi i}{m}}\right)=\sum_{k=1}^{m}\sum_{n=1}^{m}\left[\frac{n}{m}\right]e^{\frac{2\pi kni}{m}}=\sum_{n=1}^{m}\left[\frac{n}{m}\right]\sum_{k=1}^{m}e^{\frac{2\pi nki}{m}}=0$$

und also

$$H=\frac{1}{\varkappa}\prod_{n=2}^{\varphi/r}-\frac{1}{m}\sum_{k=1}^{m}F\left(e^{\frac{2k\pi i}{m}}\right)\left\{log\frac{e^{\frac{k\pi i}{m}}-e^{\frac{-k\pi i}{m}}}{i}-\frac{k\pi i}{m}\right\}\ldots\quad(9)$$

Um diese Form weiter zu vereinfachen, beweisen wir zuerst vier Hülfssätze.

§ 9. *Hülfssätze.* [1])

In allen folgenden Hülfssätzen ist das System $b_{0n}=b_{*n}=b_{1n}=\ldots=0$ ausgeschlossen.

1. $$F\left(e^{\frac{2k\pi i}{m}}\right)=(-1)^{b_{0n}+b_{1n}+\cdots}F\left(e^{\frac{2k\pi i}{m}}\right).$$

Der Beweis ist ganz analog mit dem des übereinstimmenden Satzes meiner Abhandlung in "Proceedings" Vol. XXI, S. 758.

2. Es sei $2^{h'_*}$ die höchste Potenz von 2 die auf b_{*n} teilbar ist. Wenn $b_{*n}=0$ ist, so nehme man $h'_*=h_*-2$. Ist aber auch $b_{0n}=0$ so nehme man $h'_*=h_*$. Ist $h_*=2$ so ist $h_*'=0$ zu setzen wenn $b_{0n}=1$ und $=2$ zu setzen wenn $b_{0n}=0$ ist.

Es sei weiter $l_1^{h'_1}$ die höchste Potenz von l_1 welche auf b_{1n} teilbar ist. Wenn $b_{1n}=0$ ist, so nehme man $h'_1=h_1$. u. s. w.

Es sei nun $d=2^{h_*'}l_1^{h_1'}\ldots$ so ist: $\left[\frac{n+m/d}{m}\right]=\left[\frac{n}{m}\right]$.

Beweis:

Wir faszen zuerst die Symbole

$$\left[\frac{n+m/d}{2^2}\right]^{b_{0n}}\quad\text{und}\quad\left[\frac{n}{2^2}\right]^{b_{0n}}$$

ins Auge. Ist $b_{0n}=0$ so sind sie beide $=1$, und also einander gleich. Ist $b_{0n}\neq 0$ und n gerade, so ist auch $n+\frac{m}{d}$ gerade, weil dann $\frac{m}{d}$ teilbar ist durch 4. Die Symbole sind dann beide $=0$ und

[1]) Diese Hülfssätze musz schon KUMMER benutzt haben, wiewohl in anderer Form. Die Beweise findet man aber nirgends.

wiederum gleich. Ist, zum Schlusz, $b_{0n} \neq 0$ und n ungerade, so ist auch $n + \frac{m}{d}$ ungerade und

$$\left[\frac{n + m/d}{2^2}\right]^{b_{0n}} = (-1)^{\left(n+\frac{m}{d}-1\right) b_{0n} : 2} = (-1)^{\frac{n-1}{2} b_{0n}} = \left[\frac{n}{2^2}\right]^{b_{0n}}$$

Für die übrigen Symbole kann man die Gleichheit auf analoge Art beweisen.

3. Wenn die Zahl d dieselbe Bedeutung hat wie oben, so ist $F\left(e^{\frac{2\pi ki}{m}}\right) = 0$ wenn der gröszte gemeinschaftliche Teiler von k und $m \neq d$ ist, und

$$= \left[\underset{\sim}{k/d}\right]^{-1} F\left(e^{\frac{2\pi di}{m}}\right)$$

wenn dieser gröszte gemeinschaftliche Teiler wohl $= d$ ist.

Beweis:

$$F = \sum_{n=1}^{m} \left[\underset{\sim}{n}\right] e^{\frac{2\pi nki}{m}} = \sum_{n=1}^{m} \left[\underset{\sim}{n + m/d}\right] e^{\frac{2\pi nki}{m}} =$$

$$= e^{-\frac{2k\pi i}{d}} \sum_{n=1}^{m} \left[\underset{\sim}{n + m/d}\right] e^{\frac{2\pi(n+m/d)ki}{m}}$$

Es durchläuft $n + \frac{m}{d}$ zugleich mit n alle Zahlen für welche das Symbol $\neq 0$ ist, weil zufolge der Definition der Zahl d, die Zahl m/d prim ist zu den Zahlen welche n durchläuft. Also findet man:

$$F = e^{\frac{2k\pi i}{d}} F$$

Ist k nicht durch d teilbar, so ergibt sich hieraus $F = 0$.

Nehmen wir nunmehr an dasz k durch d teilbar ist; dasz aber $k = dtv$, $m = dtm'$ so dasz dt der gröszte gemeinsame Teiler ist von k und m, $t > 1$. Man hat nun:

$$F\left(e^{\frac{2\pi ki}{m}}\right) = F\left(e^{\frac{2\pi vi}{m'}}\right) = \sum_{n=1}^{m} \left[\underset{\sim}{n}\right] e^{\frac{2\pi vni}{m'}} =$$

$$= \sum_{n=1}^{m'} e^{\frac{2\pi vni}{m'}} \sum_{s=1}^{dt} \left[\underset{\sim}{n + sm'}\right] \text{ und wegen 2:}$$

$$= d \sum_{n=1}^{m'} e^{\frac{2\pi vni}{m'}} \sum_{s=0}^{t-1} \left[\underset{\sim}{n + sm'}\right].$$

Wir werden zeigen dasz der Wert der letzten Summe Null ist. Es sei x eine Zahl der Form $1 + \frac{m'}{g} y$ (wo g der gröszte gemein-

same Teiler von n und m' ist) welche prim zu m ist. Eine solche Zahl x besteht, denn die Form enthält eine unendliche Anzahl Primzahlen.

Nun ist $nx = n\left(1 + \frac{m'}{g}y\right) = n + \frac{n}{g}ym'$ und wenn eine Zahl $n + sm'$ mit x multipliziert wird so gibt es wiederum eine Zahl derselben Form. Wäre weiter

$$x(n+sm') \equiv x(n+s'm') \left(mod \frac{m}{d}\right); \quad s, s' < t$$

so würde

$$x\,s\,m' \equiv x\,s'\,m'$$

sein, und

$$x(s - s')\,m' \equiv 0 \quad \text{also} \quad s - s' \equiv (mod\ t)$$

Dies ist unmöglich, da s und $s' < t$ sind. Es ist damit bewiesen dasz die Zahlen $n + sm'$ wo $s = 0,1, \ldots . t-1$, nach Multiplikation mit $x\left(mod \frac{m}{d}\right)$ wiederum dieselben Zahlen ergeben. Hieraus ergibt sich:

$$\left[\underset{\sim}{x}\right] \sum_{s=0}^{t-1} \left[\underset{\sim}{n + sm'}\right] = \sum_{s=0}^{t-1} \left[\underset{\sim}{x(n + sm')}\right] = \sum_{s=0}^{t-1} \left[\underset{\sim}{n+sm'}\right]$$

Also ist: $\sum_{s=0}^{t-1} \left[\underset{\sim}{n+sm'}\right] = 0.$

Um den zweiten Teil des Satzes zu beweisen, bemerken wir dasz

$$F = \left[\underset{\sim}{k/d}\right]^{-1} \sum_{n=1}^{m} \left[\underset{\sim}{n\,k/d}\right] e^{\frac{2\pi nki}{m}}$$

wenn k und m die Zahl d zum grösszten gemeinsamen Teiler haben. Die Zahl $\frac{k}{d}$ kann mit m nur diejenigen Primfaktoren gemeinsam haben, deren zugehörige Zahlen $b = 0$ sind. Denn wäre z.b. $b_{1n} \neq 0$, so ist d teilbar durch eine Potenz von l_1 welche $< l_1^{h_1}$ ist. Es ergibt sich also dasz $n\,k/d$ und n zugleich alle Zahlen durchlaufen für welche $\left[\underset{\sim}{n\,k/d}\right] \neq 0$ ist. Also ist:

$$F = \left[\underset{\sim}{k/d}\right]^{-1} \sum_{n=1}^{m} \left[\underset{\sim}{n}\right] e^{\frac{2\pi ndi}{m}} = \left[\underset{\sim}{k/d}\right]^{-1} F\left(e^{\frac{2\pi di}{m}}\right)$$

4. Wenn die Zahl d dieselbe Bedeutung hat wie oben, so ist:

$$F'\left(e^{\frac{2\pi di}{m}}\right) F\left(e^{\frac{2\pi di}{m}}\right) = (-1)^{b_{0n}+b_{1n}+\cdots}\, dm$$

F' ist dieselbe Function wie F, nachdem in letzterer die Zahlen b durch $2-b_{0n}$, $\frac{1}{2}\varphi_*-b_{*n}$, φ_1-b_{1n}, ... ersetzt sind.

Beweis: Wir faszen zuerst den Fall ins Auge wo keine der Zahlen b den Wert Null hat.

$$F = \sum_{n=1}^{m} \left[\frac{nk}{m}\right] e^{\frac{2\pi nkdi}{m}}$$

wenn k prim zu m ist, denn in diesem Falle durchläuft nk zugleich mit n ein vollständiges Restsystem $(mod\, m)$. Man folgert hieraus:

$$\left[\frac{k}{m}\right]^{-1} F = \sum_{n=1}^{m} \left[\frac{n}{m}\right] e^{\frac{2\pi nkdi}{m}}$$

Wir multiplizieren nun mit $e^{\frac{2\pi kdi}{m}}$ und nehmen die Summe über alle Werte von k welche $< m$ und prim zu m sind. Weil keine der Zahlen $b = 0$ ist, findet man:

$$F\,F' = \sum_{n=1}^{m} \left[\frac{n}{m}\right] \sum_{k} e^{\frac{2\pi(n+1)kdi}{m}} = d \sum_{n=1}^{m/d} \left[\frac{n}{m}\right] \sum_{k} e^{\frac{2\pi(n+1)kdi}{m}}$$

Es sei t der grösste gemeinschaftliche Teiler von $n+1$ und m. Man kann alle Zahlen k, welche $< m$ und prim zu m sind, $(mod\, m/d)$ verteilen in $\dfrac{\varphi(m)}{\varphi\left(\frac{m}{dt}\right)}$ Mal die Gruppe der Zahlen welche $< \dfrac{m}{dt}$ und prim zu $\dfrac{m}{dt}$ sind. So findet man:

$$F.F' = d \sum_{n=1}^{m/d} \left[\frac{n}{m}\right] \frac{\varphi(m)}{\varphi\left(\frac{m}{dt}\right)} \sum_{k} e^{\frac{2\pi(n+1)ki}{m/d}}$$

in welchem Ausdruck k nun alle Zahlen durchläuft, welche $< \dfrac{m}{dt}$ und prim zu $\dfrac{m}{dt}$ sind. Die letzte Summe ist daher die Summe der primitiven $\dfrac{m}{dt}$-te Einheitswurzeln; infolge dessen erhalten wir:

$$F\,F' = d\,\varphi(m) \sum_{n} \left[\frac{n}{m}\right] \frac{\mu\left(\frac{m}{dt}\right)}{\varphi\left(\frac{m}{dt}\right)}.$$

Nun ist $\mu\left(\dfrac{m}{dt}\right)$ nur dann nicht $= 0$, wenn $\dfrac{m}{dt}$ nicht teilbar ist durch eine Quadratzahl. Wir brauchen in obenstehender Summe also für n nur

diejenigen Werte zu nehmen, für welche $\frac{m}{dt}$ ein Teiler ist des Produktes $2\, l_1\, l_2 \ldots$. Diese Zahlen n haben die Form $-1 + st$ insoweit diese prim zu m sind, und wo s alle Werte annimmt welche $< \frac{m}{dt}$ und prim zu $\frac{m}{dt}$ sind. Setzen wir daher

$$\frac{m}{dt} = 2^a\, l_1^{a_1} \ldots ; \quad a, a_1 \ldots = 0 \text{ oder } 1$$

so ist:

$$t = 2^{h_* - h_*' - a}\, l_1^{h_1 - h_1' - a_1} \ldots$$

Zunächst nehmen wir an, die Zahl t sei durch alle Primfaktoren von m teilbar und auch durch 8. Die Exponenten, welche in obenstehender Form von t vorkommen, sind dann gröszer als Null.

Alle Zahlen $-1 + st$, wo s die oben genannten Werte annimmt, sind nun prim zu m. In obenstehender Summe kommen also $\varphi\left(\frac{m}{dt}\right)$ Glieder vor, welche $\neq 0$ sind. Diese Summe setzen wir in der folgenden Form:

$$F\,F' = (-1)^{b_{0n} + b_{1n} + \cdots}\, d\varphi\,(m) \sum_t \frac{\mu\left(\frac{m}{dt}\right)}{\varphi\left(\frac{m}{dt}\right)} \sum_s \left[\frac{1 - st}{\sim}\right].$$

Nun ist $\left[\frac{1-st}{2^2}\right] = 1$ weil t durch 8 teilbar ist.

$$\left[\frac{1 - st}{2^h}\right]^{b_{*n}} = e^{\frac{2\pi\, b_*' v_* i}{2^a}} \quad \text{denn} \quad 1 - st \equiv \pm 5^{s'} \pmod{2^{h_*}}$$

also $$1 \equiv \pm 5^{s'} \pmod{2^{h_* - h_*' - a}}$$

Da die Potenz von 2, welche in den Modul auftritt, $> 2^2$ ist, so folgt: $s' = 2^{h_* - h'_* - a - 2}\, v_*$. Die Zahl v_* ist ungerade, denn anderenfalls wäre s teilbar durch 2 und dies ist unmöglich weil t der gröszte gemeinsame Teiler von $n+1$ und m ist, und m gerade ist. Weiter hat man:

$$\left[\frac{1-st}{l_1^{h_1}}\right]^{b_{1n}} = e^{\frac{2\pi b'_{1n} v_1}{h^{a_1}}}$$

u.s.w. Aus all' diesem ersieht man dasz $\left[\frac{1-st}{\sim}\right] =$ eine primitive $\frac{m}{dt}$-te Einheitswurzel ist. Infolge dessen besteht die Gleichheit:

$$\Sigma \left[\frac{1-st}{\sim\sim}\right] = \mu\left(\frac{m}{dt}\right).$$

Folglich ist $F F' = (-1)^{b_{0n}+b_{1n}+\cdots} d\varphi(m) \sum_t \frac{1}{\varphi\left(\frac{m}{dt}\right)}$

wo t alle Teiler von $2l_1l_2\ldots$. durchläuft. Es ist leicht ersichtlich dasz φ-Mal die letzte Summe den Wert m hat, womit in unsrem besonderen Falle der Beweis erbracht ist.

Um anzugeben wie der Beweis sich gestaltet wenn nicht alle frühergemachten Annahmen erfüllt sind, fassen wir noch den Fall ins Auge wobei $h_1 - h'_1 - a_1 = 0$ und $a_1 = 1$, während alle übrigen Annahmen erfüllt sind.

In der Summe $\Sigma \left[\frac{1-st}{\sim\sim}\right]$ nimmt die Zahl s nun nicht mehr alle Werte an, welche $< \frac{m}{dt}$ und prim zu $\frac{m}{dt}$ sind; denn da t nicht durch l_1 teilbar ist, so kann $1-st$, für einige Werte von s, durch l_1 teilbar sein. Die Zahlen $z, z+l_1, \ldots, z+\left(\frac{m}{dtl_1}-1\right)l_1$ genügen der Congruenz $1-zt \equiv 0 \pmod{l_1}$. Es sind $\varphi\left(\frac{m}{dtl_1}\right)$ dieser Zahlen prim zu $\frac{m}{dt}$. In der Summe bleiben also

$$\varphi\left(\frac{m}{dt}\right) - \varphi\left(\frac{m}{dtl_1}\right) = (l_1-2)\,\varphi\left(\frac{m}{dtl_1}\right)$$

Zahlen der Form $1-st$ übrig, für welche das Symbol einen von Null verschiedenen Wert erhält. Für diese Werte von s ist ebenso wie früher:

$$\left[\frac{1-st}{2^{a_1}}\right]^{b_{0n}} = 1; \left[\frac{1-st}{2^{h_*}}\right]^{b_{*n}} = e^{\frac{2\pi b'_{*n} v_*}{2^a}}; \left[\frac{1-st}{l_2^{h_2}}\right] = e^{\frac{2\pi b'_{2n} v_2}{l_2^{a_2}}} \ldots.$$

Weiter ist aber $\left[\frac{1-st}{l_1^{h_1}}\right]^{b_{1n}} = \left[\frac{1-st+l_1}{l_1^{h_1}}\right]^{b_{1n}}$

Die Zahlen $1-st$, welche in der Summe auftreten, können wir $(\bmod l_1)$ verteilen in $\varphi\left(\frac{m}{dtl}\right)$ Mal eine Gruppe von Zahlen, welche $< l_1$ sind. Die letzte Gruppe enthält l_1-2 Glieder, unter welchen die Null und die Zahl 1 nicht vorkommen; denn keine Zahl $1-st$ ist teilbar durch l_1 und t und s sind nicht teilbar durch l_1, weil s

prim zu $\frac{m}{dt}$ ist und diese letzte Zahl teilbar ist durch l_1. Wir fügen nun die Glieder, in welchen das Symbol $\left[\frac{1-st}{l_1^{h_1}}\right]^{b_{1n}}$ denselben Wert hat, zusammen.

Es wird dieser Wert dann multiplizirt mit der Summe der primitiven $\frac{m}{dt\,l_1}$-ten Einheitswurzeln. Wir sind also zu der folgenden Gleichheit gelangt:

$$\Sigma\left[\frac{1-st}{\sim}\right]=\mu\left(\frac{m}{dt\,l_1}\right)\times\sum_{n=2}^{l_1}\left[\frac{n}{l_1^{h_1}}\right]^{b_{1n}}.$$

Die letzte Summe hat den Wert -1, wie sich aus dem ersten Hülfssatz dieses Kap. ergibt, denn es ist:

$$0=\sum_{u=1}^{l_1^{h_1}}\left[\frac{n}{\sim}\right]=(l_1^{h_1-1}-1)\sum_{n=1}^{l_1}\left[\frac{n}{l_1^{h_1}}\right]^{b_{1n}}$$

da, weil b_{1n} durch $l_1^{h_1-1}$ teilbar ist:

$$\left[\frac{n}{l_1^{h_1}}\right]^{b_{1n}}=\left[\frac{n'}{l_1^{h_1}}\right]^{b_{1n}}\quad\text{wenn}\quad n\equiv n'\,(mod\,l_1)$$

Schliesslich: $\Sigma\left[\frac{1-st}{\sim}\right]=-1\,.\,\mu\left(\frac{m}{dt\,l_1}\right)=\mu\left(\frac{m}{dt}\right).$

Der Beweis gestaltet sich weiter wie im vorigen Falle. Wir haben nun hinreichend angegeben wie der Beweis erbracht wird wenn man noch andere der früher gemachten Einschränkungen aufhebt; nur den Fall $b_{1n}=0$ werden wir noch weiter untersuchen.

In der Summe F haben in diesem Falle alle Glieder, für welche n prim zu $\frac{m}{l_1^{h_1}}$ ist, einen von Null verschiedenen Wert. Und weiter ist

$$\left[\frac{n}{\sim}\right]=\left[\frac{n'}{\sim}\right]\quad\text{wenn}\quad n\equiv n'\left(mod\,\frac{m}{l_1^{h_1}}\right).$$

Auszerdem kann man alle Zahlen, welche $<m$ und prim zu m sind, $\left(mod\,\frac{m}{l_1^{h_1}}\right)$ verteilen in $l_1^{h_1}$ Mal die Gruppe der Zahlen welche $<\frac{m}{l_1^{h_1}}$ und prim zu $\frac{m}{l_1^{h_1}}$ sind. Also:

$$F=\sum_n\left[\frac{n}{\sim}\right]\left(e^{\frac{2\pi n\,di}{m}}+e^{\frac{2\pi\left(n+\frac{m}{l_1^{h_1}}\right)di}{m}}+\cdots\right)$$

Da d teilbar ist durch $l_1^{h_1}$, ergibt sich hieraus

$$F = l_1^{h_1} \sum_n \left[\frac{n}{\quad} \right] e^{\frac{2\pi n di}{m}} = l_1^{h_1} F_1 \left(e^{\frac{2\pi n di}{m}} \right)$$

Die Function F_1 leitet man aus F ab, indem man $\frac{m}{l_1^{h_1}}$ anstatt m nimmt. Für die Function F_1 haben wir schon bewiesen:

$$F F' = l_1^{h_1} F_1 . l_1^{h_1} F_1' = l_1^{2h_1} F_1 F_1 =$$

$$= (-1)^{b_{0n} + b_{1n} + \cdots} l_1^{2h_1} \frac{d}{l_1^{h_1}} \cdot \frac{m}{l_1^{h_1}} = (-1)^{b_{0n} + b_{1n} + \cdots} dm.$$

Damit ist der Beweis in unserem Falle wiederum erbracht.

§ 10. *Sätze über die Realität des Unterkörpers k und Bestimmung der Zahl $\varkappa$.*

1. Wenn die Summe $b_{0i} + b_{1n} + \ldots.$, für alle Werte von n, gerade ist, so ist der Unterkörper k reell und andernfalls imaginär.

Beweis: Die den Körper k bestimmende Zahl ist

$$\eta = Z^{A^{(1)}} + Z^{A^{(2)}} + \ldots. + Z^{A^{(r)}} \text{ [1]}$$

wo $A^{(i)}$ jede Zahl der Untergruppe g bedeutet.

Nun ist

$$A^{(i)} \equiv A_0^{a_{0i}} A_*^{a_{*i}} A_1^{a_{1i}} \ldots. \pmod m$$

wo die Exponenten den Congruenzen (3) genügen. Aus der Annahme dasz alle Summen $b_{0n} + b_{1n} + \ldots$ gerade sind, folgt dasz das Wertsystem $a_0 = 1$; $a_* = 0$; $a_1 = \frac{1}{2}\varphi_1$; ... den Congruenzen (3) genügt. Dann genügt aber auch das System

$$a_{0i} + 1, a_{*i}, a_{1i} + \tfrac{1}{2}\varphi_1, \ldots.$$

Nun ist noch

$$A_0^{a_{0i}+1} A_*^{a_{*i}} A_1^{a_{1i}+\frac{1}{2}\varphi_1} \ldots. \equiv -A^{(i)} \pmod m$$

Die Zahlen $A^{(i)}$ en $-A^{(i)}$ sind $(\mathit{mod}\ m)$ verschieden, denn wäre

$$A^{(i)} \equiv -A^{(i} \pmod m$$

so würde

$$2A^{(i)} \equiv 0 \pmod m$$

sein, und dies ist unmöglich, weil die $A^{(i)}$ prim zu m sind. Wir haben also bewiesen dasz die Zahlen $A^{(i)}$ von g in Paaren verteilt werden können. Für jedes Paar hat $Z^{A^{(i)}} + Z^{-A^{(i)}}$ einen reellen Wert. Man folgert hieraus leicht dasz η reell ist.

Nun musz noch der Beweis des zweiten Teils des Satzes erbracht werden.

Alle reellen Zahlen des Kreiskörpers K bleiben unverändert für

[1]) „W.", S. 85.

die Substitution $s = (Z : Z^{-1})$. Denn es sei Ω eine reelle ganze Zahl, so ist

$$\Omega = a_0 + a_1 Z + \ldots .$$

$$a_1 \sin\frac{2\pi}{m} + a_2 \frac{\sin 4\pi}{m} + \ldots . = 0$$

und

$$s\,\Omega = a_0 + a_1 Z^{-1} + \ldots . \quad = a_0 + a_1 \cos\frac{2\pi}{m} + \ldots .$$

Also:

$$\Omega - s\,\Omega = 0.$$

Ebensoleicht beweist man, umgekehrt, dasz eine jede Zahl, welche durch die Substitution s nicht geändert wird, eine reelle Zahl ist. Man ersieht hieraus dasz jeder reelle Unterkörper von K ein Unterkörper ist des zu der Gruppe s, s^2 gehörenden Unterkörpers und dasz ein Unterkörper, der zu einer Untergruppe gehört, welche die Substitution s nicht enthält, imaginär ist. Wenn nun nicht alle Summen $b_{0n} + b_{1n} + \ldots .$ gerade sind, so genügt das System $a_0 = 1$; $a_* = 0$; $a_1 = \frac{1}{2}\varphi_1 \ldots .$ den Congruenzen (3) nicht; die Untergruppe g kann daher in diesem Falle die Substitution s nicht enthalten. Der Körper k ist imaginär.

2. Wenn die Summe $b_{0n} + b_{1n} + \ldots .$ nicht für alle Werte von n einen geraden Wert hat, so ist die Anzahl der Systeme der b, für welche dieser Wert ungerade ist, gleich $\frac{\varphi}{2r}$.

Beweis: Alle Systeme der b für welche die Summe gerade ist, bilden eine Gruppe, da $2, \varphi_1, \varphi_2, \ldots .$ gerade sind. Um aus dieser Gruppe, die Gruppe aller Systeme der b zu bekommen, musz man die erstgenannte Gruppe multiplizieren mit einer Gruppe in welcher jedes System der b (ausgenommen die identische Substitution $b_{01} = 0$, $b_{*1} = 0, \ldots .$) eine ungerade Summe besitzt. Wenn, in dieser letzten Gruppe, sich zwei Systeme der b vorfänden mit ungerader Summe, so würde das System, dasz man durch Aufzählung der übereinstimmenden Zahlen dieser beiden Systemen erhielt, eine gerade Summe $b_{0n} + b_{1n} + \ldots .$ haben. Ein solches System kann aber in dieser Gruppe nicht auftreten. In der Gruppe können also nicht zwei Systeme vorkommen (ausser $b_{01} = b_{*1} = \ldots . = 0$). Die Gruppe hat daher den Grad 2.

Es gibt also $\frac{1}{2} \cdot \frac{\varphi}{r}$ Systeme der b, welche eine gerade Summe haben und natürlich gleichviel mit ungerader Summe.

Bestimmung der Zahl $\varkappa$ [1]).

[1]) „H.", S. 229.

1°. Wenn der Körper k reell ist, so ist $w = 2$ denn ± 1 sind die einzigen reellen Einheitswurzeln.

Es ist weiter $r_2 = 0$ und $r_1 = \frac{\varphi}{r}$ weil der Körper ein GALOIS'scher ist.

2°. Wenn der Körper k imaginär ist, so ist $r_1 = 0$ und $r_2 = \frac{\varphi}{2r}$. Nun musz noch die Zahl w bestimmt werden.

Es sei $w_0 = 1$ wenn alle Zahlen $a_0 = 0$ sind; andernfalls $w_0 = 0$.

Es sei 2^{w_*} die höchste Potenz von 2 welche auf alle Zahlen a_* teilbar ist und $w_* = h_* - 2$ wenn alle Zahlen $a_* = 0$. $u_* = 0$ wenn nicht alle Zahlen a_* durch 2 teilbar sind und auch $= 0$ wenn nicht alle $a_0 = 0$ sind; $u_* = 1$ wenn alle $a_0 = 0$ und alle a_* durch 2 teilbar sind.

Es sei $l_1^{w_1}$ die höchste Potenz von l_1 welche auf alle Zahlen a_1 teilbar ist, und $w_1 = h_1 - 1$ wenn alle $a_1 = 0$ sind. $u_1 = 0$ wenn nicht alle Zahlen a_1 durch $l_1 - 1$ teilbar sind und $u_1 = 1$ wenn dies wohl so ist. U. s. w.

Dann ist:

$$\text{wenn } h_* \geqq 3 \text{ ist: } w = 2^{w_0+w_* u_*+1} l_1^{u_1(w_1+1)} \ldots .$$

$$\text{wenn } h_* = 2 \text{ ist: } w = 2^{w_0+1} l_1^{u_1(w_1+1)} \ldots .$$

$$\text{wenn } h_* = 0 \text{ ist: } w = 2 l_1^{u_1(w_1+1)} \ldots .$$

Beweis: Der Körper k kann nur diejenigen Einheitswurzeln enthalten, welche Potenzen von Z sind, da der Kreiskörper K nur solche enthält. Nur wenn m ungerade ist, enthält K auch die Potenzen von Z mit negativem Vorzeichen. Nehmen wir nun an dasz Z^a in k liegt, dann bleibt diese Zahl ungeändert für die Substitutionen von g. Wenn wir also alle Zahlen von g durch $A^{(i)}$ darstellen, so ist

$$Z^{aA^{(i)}} = Z^a$$

woraus sich ergibt:

$$a(A^{(i)} - 1) \equiv (\mathit{mod}\ m) \quad \ldots\ldots\ldots \quad (10)$$

Wenn, umgekehrt, die Zahl a dieser Congruenz genügt, für alle Zahlen $A^{(i)}$ von g, so enthält der Körper k die Einheitswurzel Z^a.

Sind alle Zahlen $a_0 = 0$ so enthält der Körper die Einheitswurzeln $\pm i$.

Beweis: Es gilt für jede Zahl $A^{(i)}$:

$$A^{(i)} \equiv 5^{a_*} (\mathit{mod}\ 2^h) \quad \text{also} \quad A^{(i)} \equiv 1 (\mathit{mod}\ 4).$$

Aus (10) ergibt sich daher $a = \frac{m}{4}$, d. h. die Einheitswurzel $Z^{\frac{m}{4}} = i$ liegt im Körper k. Daher auch $-i$.

Sind alle Zahlen $a_* = 0$ und alle Zahlen $a_0 = 0$ so enthält k die Einheitswurzel $e^{\frac{2\pi i}{h_*}}$. Man beweist dies auf dieselbe Weise.

Sind alle Zahlen $a_0 = 0$ und alle Zahlen a_* höchstens teilbar durch 2^{w_*} so enthält k die Einheitswurzel $e^{\frac{2\pi i}{2^{w_*+2}}}$. U.s.w.

Alle Beweise werden mit Hülfe der Congruenz (10) erbracht.

Man findet nun leicht die im Satze genannte Formel wenn man beachtet dasz, wenn der Körper die Einheitswurzel $e^{\frac{2\pi i}{h_1^{w_1+1}}}$ enthält, auch die $h_1^{w_1+1}$ ersten Potenzen dieser Zahl im Körper liegen.

§ 11. *Ableitung des entgültigen Ausdrucks für die Klassenzahl.*

Satz: Wenn, für jedes System der b, die Summe $b_{0n} + b_{1n} + \ldots$ gerade ist, wenn also der Körper k reell ist, so stellt sich die Klassenanzahl H dieses Körpers, wie folgt, dar:

$$H = \frac{\prod_{n=2}^{\varphi/r} \sum_{s=1}^{<\frac{m}{2}} \left[\overbrace{b_{0n}, b_{*n}, b_{1n} \ldots}^{s} \right] log\, A_s}{R}$$

Hierin ist $A_s = \sqrt{(1-Z^s)(1-Z^{-s})}$ und R der Regulator. Auszerdem ist angenommen: $b_{01} = b_{*1} = b_{11} = \ldots = 0$.

Ist $h_* = 2$ so musz b_{*n} weggelassen werden und ist $h_* = 0$ so musz auch b_{0n} weggelassen werden.

Wenn die Summe $b_{0n} + b_{1n} + \ldots$ nicht für jedes System der b gerade ist, wenn also der Körper k imaginär ist, so stellt sich die Klassenanzahl dieses Körpers k, wie folgt, dar:

$$H = \frac{w \prod_{n}^{} \sum_{s=1}^{m} \left[\overbrace{b_{0n}, b_{*n}, b_{1n}, \ldots}^{s} \right] s}{(2m)^{\varphi/2r}} \cdot \frac{\prod_{n}^{} \sum_{s=1}^{<\frac{m}{2}} \left[\overbrace{b_{0n}, b_{*n}, b_{1n}, \ldots}^{s} \right] log\, A_s}{R'}$$

Hierin ist w die in § 10 bestimmte Zahl. Das erste Produkt ist über alle Werte von n, für welche die Summe $b_{0n} + b_{1n} + \ldots$ ungerade ist, zu erstrecken; das zweite Produkt über alle Werte von n für welche diese Summe gerade ist.

Weiter ist $A_s = \sqrt{(1-Z^s)(1-Z^{-s})}$ und R' die Determinante der Logarithmen der absoluten Werte eines Systems von Grundeinheiten. Ist $h_* = 2$ so ist b_{*n} wegzulassen, und ist $h_* = 0$ so ist auch b_{0n} wegzulassen.

Beweis:

Zür Ableitung des ersten Ausdrucks benutzen wir (9) und ersetzen, in der darin auftretenden Summe, die Grösze k durch $m-k$. Wenn

wir dann den ersten Hülfssatz von § 9 benutzen, so finden wir $\Sigma F . \frac{k\pi i}{m} = 0$. Die Gleichheit (9) läszt sich daher umformen zu:

$$H = \frac{1}{\varkappa} \sum_{n=2}^{\varphi/r} - \frac{1}{m} \sum_{k=1}^{m} F\left(e^{\frac{2\pi ki}{m}}\right) log\, A_k$$

wenn:

$$A_k = \frac{e^{\frac{k\pi i}{m}} - e^{-\frac{k\pi i}{m}}}{i} = ie^{-\frac{k\pi i}{m}}\left(1 - e^{\frac{2k\pi i}{m}}\right) =$$

$$= -ie^{\frac{k\pi i}{m}}\left(1 - e^{-\frac{2k\pi i}{m}}\right) = \sqrt{\left(1 - e^{\frac{2k\pi i}{m}}\right)\left(1 - e^{-\frac{2k\pi i}{m}}\right)}$$

Dieses Resultat für H läszt sich noch veränderen mittels Hülfssatz 3 und 4 von § 9. Aus Hülfssatz (3) ergibt sich, dasz in der Summe nur diejenigen Glieder übrig bleiben für welche die Zahl k mit m den grösten gemeinsamen Teiler d hat. Diese Zahl d deuten wir weiter an durch d_n. Die Summe wird zu

$$F\left(e^{\frac{2\pi d_n i}{m}}\right) \sum_{k} \left[\frac{k/d}{\sim\sim}\right]^{-1} log\, A_k$$

Es sei nun $k = k' d_n$, so ist, wie man leicht findet:

$$A_k = A_{k'} . A_{k' + \frac{m}{d_n}} \ldots . A_{k' + (d_n - 1)\frac{m}{d_n}}$$

Die Summe wird daher zu:

$$F\left(e^{\frac{2\pi d_n i}{m}}\right) \sum_{k} \left[\frac{k/d_n}{\sim\sim}\right]^{-1} (log\, A_{k'} + log\, A_{k' + \frac{m}{d_n}} + \ldots .) \quad . \quad (11a)$$

und durch Hülfssatz 2 § 9 zu:

$$F\left(e^{\frac{2\pi d_n i}{m}}\right) \sum_{s=1}^{m} \left[\frac{s}{\sim\sim}\right]^{-1} log\, A_s \quad . \quad . \quad . \quad . \quad . \quad (11)$$

denn, weil k' prim zu m/d_n ist und in (11a) die Zahl k alle Werte annimmt die mit m die Zahl d_n zum grösten gemeinsamen Teiler haben, bekommt man eine Summe, in welcher s alle Zahlen $< m$ durchläuft, welche prim zu m sind oder nur noch teilbar durch diejenigen in m aufgehenden Primzahlen, deren betreffende $b = 0$ sind. s durchläuft also alle Zahlen für welche $\left[\frac{s}{\sim\sim}\right] \neq 0$ ist.

In der Formel für H kommt nun das Produkt

$$\prod_{n=2}^{\varphi/r} F\left(e^{\frac{2\pi d_n i}{m}}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

zum Vorschein. Um dessen Wert zu berechnen benutzen wir Hülfs-

satz (4) von § 9. Da zu jeder Substitution einer Gruppe auch die reciproke Substitution vorkommt, so wird ins Besondere hier zu jedem System $b_{0n}, b_{*n}, \ldots$ auch das System $2-b_{0n}, \frac{1}{2}\varphi_* - b_{*n}, \varphi_1 - b_{1n} \ldots$ auftreten. Wenn diese Systeme von einander verschieden sind, so ist, nach dem ebengenannten Hülfssatz, das Produkt der zugehörigen Functionen F und F' gleich $d_n m$. Sind sie aber einander gleich so hat das Produkt den Wert $\pm\sqrt{d_n m}$.

Das Produkt (12) hat daher den Wert

$$m^{\frac{\varphi}{2r}-1} \prod_{n=2}^{m} \sqrt{d_n} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

wenn man das Vorzeichen ausser Acht läszt.

Bevor wir nun den Wert dieses Produktes entgültig bestimmen, formen wir die Summe aus (11) um. Zuerst können wir darin den Exponent -1 weglassen, weil, wie schon bemerkt worden ist, zu jedem System der Zahlen b, auch das System $2-b_{0n}, \ldots$ vorkommt.

Weiter ist

$$\sum_{s=1}^{m} \left[\frac{s}{m}\right] \log A_s = \sum_{s=1}^{<\frac{m}{2}} + \sum_{s=1}^{<\frac{m}{2}} \left[\frac{m-s}{m}\right] \log A_{m-s}$$

Denn, für den Fall dasz m ungerade ist, folgt diese Gleichheit aus der bloszen Bemerkung dasz $\left[\frac{m}{m}\right] = 0$ ist. Ist, im Gegensatz, m gerade, so ist offenbar $\left[\frac{m/2}{m}\right] = 0$ da m durch 4 teilbar ist.

Zur weiteren Umformung benutzen wir die leicht zu erhaltende Beziehung:

$$\left[\frac{-s}{m}\right] = (-1)^{b_{0n}+b_{1n}+\cdots} \left[\frac{s}{m}\right]$$

und auch die Gleichheit $A_{m-s} = A_s$, und erhalten so für die Summe:

$$2 \sum_{s=1}^{<m:2} \left[\frac{s}{m}\right] \log A_s$$

Nach Einführung in die Formel (9) und nach Einsetzung des Wertes von $\varkappa$ und der in § 6 berechneten Discriminante d, erhält man den im Satze angegebenen Endausdruck für die Klassenzahl, nachdem das Produkt aller Zahlen d_n welches in (13) auftritt berechnet ist. Weil die Berechnung dieses Produktes dieselbe ist beim zweiten Teil des Satzes, so werden wir diese Berechnung bis zum Ende des Beweises aufschieben.

Beweis des zweiten Teils des Satzes.

Das Produkt aus (9) zerlegen wir in zwei Produkten: das erste

ist zu erstrecken über alle Werte von n für welche $b_{0n} + b_{1n} + \ldots\ldots$ ungerade ist, während das zweite zu erstrecken ist über alle Werte von n für welche diese Summe gerade ist. Das letzte Produkt läszt sich auf dieselbe Weise umformen wie beim Beweise des ersten Teils des Satzes. Beim ersten Produkt erhalten wir, durch Benutzung des ersten Hülfssatzes § 9, nachdem in (9) $m-k$ statt k eingeführt ist:

$$\sum_{k=1}^{m} F\left(e^{\frac{2\pi ki}{m}}\right) log\, A_k = 0.$$

Es bleibt von der Summe aus (9) also nur übrig

$$-\frac{\pi i}{m} \sum_{k=1}^{m} k\, F\left(e^{\frac{2\pi ki}{m}}\right).$$

Wegen Hülfssatz 3 (§ 9) wird dies zu

$$-\frac{\pi i}{m} F\left(e^{\frac{2\pi d_n i}{m}}\right) \sum_{k} k \left[\frac{k/d}{\sim}\right]^{-1} = -\frac{\pi\, di}{m} F \cdot \sum_{k} \frac{k}{d} \left[\frac{k/d}{\sim}\right]^{-1}$$

wo k über alle Zahlen zu erstrecken ist, die mit m die Zahl d_n zum grösten gemeinsamen Teiler haben.

Weiter ist, wenn $k = k' d_n$; $d = d_n$;:

$$d_n \sum_{k} \frac{k}{d} \left[\frac{k/d}{\sim}\right]^{-1} = \sum_{k'} k' \left[\frac{k'}{\sim}\right]^{-1} + \sum_{k'} (k' + m/d) \left[\frac{k' + m/d}{\sim\sim}\right]^{-1} +$$

$$\ldots\ldots \sum_{k'} (k' + (d-1)\, m/d) \left[\frac{k' + (d-1)\, m/d}{\sim\sim}\right]^{-1}$$

da

$$\Sigma \left[\frac{k'}{\sim}\right]^{-1} + \Sigma \left[\frac{k' + m/d}{\sim\sim}\right]^{-1} + \ldots = \sum_{s} \left[\frac{s}{\sim}\right] = 0$$

ist wegen Hülfssatz 1 von § 8. Wir erhalten also das Resultat:

$$-\frac{\pi i}{m} F\left(e^{\frac{2\pi d_n i}{m}}\right) \sum_{s=1}^{m} \left[\frac{s}{\sim}\right]^{-1}_{s}$$

in welchem der Exponent -1 darf weggelassen werden da, wenn die Summe $b_{0n} + b_{1n} + \ldots\ldots$ ungerade ist, auch die Summe der Werte $2-b_{0n}$, $\frac{1}{2}\varphi_* - b_{*n}, \ldots\ldots$ ungerade ist.

Es ist ersichtlich dasz in (9) wiederum das Produkt $\sqrt{\prod_{n=2} d_n}$ zum Vorschein kommt, wenn wir die Faktoren zu zweien nehmen und jedes Produkt mittels Hülfssatz 4 (§ 9) umformen. Dabei erscheinen dann Potenzen von i, und ebensolche treten auch bei der weiteren Umformung noch auf. Man kann sie jedoch auszer Acht lassen weil die Zahl H natürlich eine positive ganze Zahl ist.

§ 12. *Berechnung des Wertes des Produktes* $\prod_{n=2}^{\varphi/r} d_n$.

Wir bestimmen die Potenz von 2 und die von l_1, welche in dem Produkte vorkommen.

Es gibt $\frac{1}{2}\varphi_*$ voneinander verschiedene Zahlen b_{*n} da die Untergruppe g primär ist. Jeder dieser Werte kommt also in allen Systeme der b genau $\frac{\varphi}{r} : \frac{1}{2}\varphi_*$ Mal vor. Nun gibt es unter den voneinander verschiedenen Werten der b_{*n} genau.

2^{h_*-4} Zahlen welche genau durch 2 teilbar sind.

2^{h_*-5} „ „ „ „ 2^2 „ „

. . . . . .

2^1 „ „ „ „ 2^{h_*-3} „ „

1 „ „ „ „ 2^{h_*-2} „ „

Alles zusammen genommen gibt dies die folgende Potenz von 2:

$$2^{(2^{h_*-2}-1)\frac{2\varphi}{\varphi_* r}-(h_*-2)}$$

wenn man bedenkt dasz das System $b_{0n} = b_{*n} \ldots = 0$ nicht mitgezählt werden musz. Man musz nun noch Rücksicht nehmen auf den Fällen worein b_{0n} und b_{*n} beide zugleich den Wert Null annehmen. Für einen jeden solchen Fall musz obenstehende Potenz von 2 noch mit 2^2 multipliziert werden. Nun gibt es $\frac{\varphi}{2r}$ Systeme in welchen $b_{0n} = 0$ ist. Die Systeme in welchen $b_{0n} = b_{*n} = 0$ ist, stellen eine Untergruppe der Gruppe aller Systeme b dar. Die Systeme, in welchen $b_* = 0$ ist, stellen auch eine Untergruppe dar, welche die erstgenannte Untergruppe enthält. Diese erstgenannte Untergruppe musz, wie leicht ersichtlich, multipliziert werden mit der Gruppe des Grades 2:

$$b_{0n} = 0; \quad b_{*n} = 0; \quad b_{1n} = 0, \ldots .$$
$$b_{0n} = 1; \quad b_{*n} = 0; \quad b_{1n} = 0, \ldots .$$

um die Gruppe der Systeme, in welchen $b_{*n} = 0$ ist, zu erhalten. Man erkennt also dasz die Hälfte der Anzahl der Systeme in welchen $b_{*n} = 0$ ist, die Anzahl der Systeme ist, in welchen $b_{0n} = b_{*n} = 0$ ist. Dies ist also der Fall wenn nicht für jeden Wert von n $b_{0n} = b_{*n} = 0$ ist. Ist jedoch $b_{0n} + b_{*n}$ für alle Werte von n gerade, so ist immer $b_{0n} = b_{*n} = 0$.

Es ist nun leicht ersichtlich dasz obenstehende Potenz von 2 noch multipliziert werden musz mit

$$2^{2\left(\frac{(t+1)\varphi}{\varphi_* r}-1\right)}$$

wo t dieselbe Bedeutung hat wie in § 6 Satz 8.

Um die Potenz von l_1 zu bestimmen führen wir das Folgende an: Weil d_1 [1]) der gröszte gemeinsame Teiler aller Zahlen b_{1n} ist, so gibt es $\frac{\varphi_1}{d_1}$ voneinander verschiedene Zahlen b_{1n}. Ein jeder solchen Wert kommt daher $\frac{\varphi}{r} : \frac{\varphi_1}{d_1}$ Mal vor. Weil g primär ist, ist d_1 nicht durch l_1 teilbar, und, wenn $h_1 = 1$ nicht durch $l_1 - 1$. Die durch l_1 teilbaren Zahlen b_{1n} sind daher

$$d_1 l_1 \,,\, 2\, d_1 l_1 \,, \ldots . \frac{l_1 - 1}{d_1} \,.\, d_1 l_1^{h_1 - 1}$$

Von diesen sind

$\frac{(l_1-1)^2}{d_1} l_1^{h_1-3}$ Zahlen genau durch l_1 teilbar.

$\frac{(l_1-1)^2}{d_1} l_1^{h_1-4}$ „ „ „ l_1^2 „

, ...

$\frac{l_1-1}{d_1} - 1$ Zahlen genau durch $l_1^{h_1-1}$ teilbar.

Wenn man noch Rücksicht nimmt auf den Fall wo alle $b = 0$ sind, so erhält man für die Potenz von l_1 welche im Produkte aller d_n vorkommt:

$$l_1^{\left(\frac{l^{h_1-1}-1}{d_1}\right)\frac{\varphi d_1}{\varphi_1 r} - h_1}$$

Wenn nun D die in § 6 bestimmte Discriminante des Körpers k darstellt, so gibt eine leichte Rechnung allgemein die Formel

$$\sqrt{D} \,.\, \prod_{n=2}^{\varphi/r} d_n = m^{\frac{\varphi}{2r} - \frac{1}{2}} \,.$$

Im Nenner des Ausdrucks für H findet sich nun noch der Regulator R. Nehmen wir ein System Grundeinheiten η_i, so gilt:

$$log\, |\eta_i^{(k)}| = \tfrac{1}{2}\, l_k(\eta_i) \text{ [2])}$$

weil die conjugirten Körper reell sind. Nach Einführung in R kann man die Faktoren 2 aus dem Determinante herausheben, wonach die Richtigheit der im Satz § 11 angegebene zweite Formel bewiesen ist.

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[1]) Diese Zahl d_1 hat also dieselbe Bedeutung wie in § 4.2. Eine Verwirrung mit den in § 9.2 bestimmten Zahl d, welche im Anfang des Beweises von § 11 d_n genannt ist, ist wohl nicht zu befürchten.

[2]) „H." S. 215. Im Ausdruck auf S. 376 (oben) nennt Herr Hilbert die Zahl R den Regulator. Dies ist nicht in Ubereinstimmung mit seiner Definition auf Seite 221.

Chemistry. — *"On the preparation of α-sulphopropionic acid."* By H. J. BACKER and J. V. DUBSKY. (Communicated by Prof. F. M. JAEGER.)

(Communicated in the meeting of September 27, 1919).

The only practical method for the preparation of the α-sulphopropionic acid has hitherto been the one found by FRANCHIMONT [1]) i. e, the action of sulphuric acid with propionic acid anhydride.

Besides FRANCHIMONT's general method, an analogical method has been used by MELSENS [2]) for the preparation of the sulphonacetic acid, the first term of the sulphocarbonic acids, namely the action of sulphuric acid anhydride with acetic acid.

These two methods show much similarity. In both cases the mixed anhydride, the acetylsulphuric acid, is formed as an intermediary product, as FRANCHIMONT [3]) had already supposed and STILLICH [4]) afterwards established, and as VAN PESKI [5]) proved for the reaction of MELSENS.

With a view to comparing it with the method of FRANCHIMONT we have applied MELSENS' method also to propionic acid, by treating it with sulphurtrioxide.

In both cases the reaction turned out, just as in the case of the acetic acid, to be indirect, while the mixed anhydride, the propionylsulphuric acid, must be taken to be the intermediary product.

$$\left.\begin{aligned} CH_3.CH_2.CO_2H + SO_3 &= CH_3.CH_2.CO_2SO_3H \rightleftharpoons \\ &= CH_3.CH(SO_3H).CO_2H. \end{aligned}\right\} \text{I.}$$

$$\left.\begin{aligned} (CH_3.CH_2.CO)_2O + H_2SO_4 &= \\ = CH_3.CH_2.CO_2H + CH_3.CH_2.CO_2SO_3H &\rightarrow CH_3.CH(SO_3H).CO_2H. \end{aligned}\right\} \text{II.}$$

When the substances are mixed carefully whilst cooling, a colourless, very viscous liquid is formed, which yields the sulphopropionic acid only at a higher temperature, with development of heat and brown coloration.

We were able to follow the process of this reaction by titration,

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1) Recueil trav. chim. **7**, 27 (1888).
2) Ann. der Chemie **52**, 276 (1844).
3) Versl. dezer Akad. **16**, 373 (1881).
4) Ber. d. dtsch. chem. Ges. **38**, 1241 (1905).
5) Versl. dezer Akad. **22**, 996 (1914).

as the propionylsulphuric acid, when hydrolyzed, neutralizes three equivalents of a basis, and the sulphonic acid only two equivalents.

The sulphopropionic acid was separated and weighed in the form of its baryumsalt. From 1 molecule propionic acid with sulphurtrioxide we got an average yield of 0.35 mol. sulphopropionic acid, and from 1 mol. propionic acid anhydride with sulphuric acid (monohydrate) 0.55 mol.

Compared with FRANCHIMONT's method, the reaction with sulphurtrioxide is experimentally less simple. The reaction is violent, the product is more coloured, the baryumsalt likewise, and the yield is various. Only for the preparation of large quantities may the reaction be recommended.

FRANCHIMONT's reaction, however, is very easy to carry out; but it is a drawback, that only half the propionic acid anhydride can be transformed into the sulphonic acid, the other half returning to propionic acid.

We have therefore tried to combine the advantages of both methods, the better yield of the first and the greater purity of the product obtained by the second.

For this purpose, it is possible to mix propionic acid anhydride with sulphuric acid, and to treat the product, containing one molecule free propionic acid, with the equimolecular quantity of sulphurtrioxide.

However, it is simpler to let the propionic acid anhydride react directly with *pyrosulphuric acid.*

$$(CH_3 . CO)_2O + H_2S_2O_7 = 2\,CH_3 . CH(SO_3H) . CO_2H.$$

By this method we got an average yield of 0.75 mol. sulphonic acid from 1 mol. propionic acid anhydride.

As the pyrosulphuric acid may be added in the crystallized state to the propionic acid, the method is easier than the reaction with sulphurtrioxyde. The reaction proceeds more quietly and the product is purer.

We have applied this method to acetic acid anhydride also, but for such an easily accessible substance, the original method of FRANCHIMONT is simpler.

For the sulphuration of precious carbonic acid anhydrides, the method just described may, however, be recommended for its higher yield.

Groningen, Sept. 8, 1919. *Org. Chem Lab. of the Univ.*

Mathematics. — *"Some applications of the quasi-uniform convergence on sequences of real and of holomorphic functions"*. By Prof. J. WOLFF. (Communicated by Prof. L. E. J. BROUWER).

(Communicated in the meeting of September 27, 1919).

We consider a region G of the complex plane, and a sequence of functions $f_1, f_2, \ldots$ which are all analytical within G. If the sequence converges uniformly in any closed region within G, then the limiting function f is analytical within G. This theorem however, enunciated by WEIERSTRASZ, has of late been considerably extended. If it is known that the sequence converges at the points of a set having a point of the interior of G for a limiting point, and besides, that the sequence is uniformly limited in every closed region internal to G, then this is already sufficient to conclude to the uniform convergence of the sequence in any closed region within G, which involves that the limiting function f is analytical [1]). This same conclusion may be drawn, G being a circle, if only the sequence of functions is supposed to be uniformly limited within G and convergent at the points $z_1, z_2, \ldots$, such that $\prod_1^\infty (z_k - z_0) = 0$, where z_0 is the centre of G and $z_k \neq z_0$ [2]); also if it is only supposed that there exist two definite numbers a and b such that nowhere in the interior of G one of these two values is assumed by a function of the sequence, and besides that the sequence converges at the points of a set having a point internal to G for a limiting point [3]). None of these theorems lead to other sequences of functions than those which are embraced already by WEIERSTRASZ' theorem. The question now could be raised whether it is possible that a sequence of functions, each of which is analytical within a region G, may converge to a function, analytical within G, without this convergence being uniform in every closed region contained in G. That such sequences actually exist is shown by an example,

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[1]) G. VITALI. Annali di Matematica, Serie 3a, tomo 10 (1904), p. 65. For a simple proof *vide* a.o. Verh. der Kon. Ac. v. W., dl. 27 (1918), p. 319.

[2]) W. BLASCHKE. Leipz. Berichte, Band 67 (1915), p. 194.

[3]) C. CARATHÉODORY and E. LANDAU. Sitz. Ber. Ak. v. W. Berlin, Band 32 (1911), p. 587.

given by MONTEL with the purpose of refuting the assertion of POMPEJU that it should be necessary — in order to render the limiting function f of a within G convergent series of analytical functions analytical in G — to suppose that it is permitted to integrate the series termwise along every curve of integration in G[1]). In fact MONTEL's series converges to zero throughout the whole plane whilst the termwise integration along the real axis from 0 to 1 yields not 0 but 1, so a uniform convergence in every limited region is here altogether out of the question. We state, besides, that it is neither permitted to differentiate MONTEL's series termwise for $z = 0$, as this would yield the result $+\infty$ instead of zero. The example of MONTEL is embraced by none of the foregoing theorems. The want is now felt of a necessary and sufficient condition that a sequence of analytical functions, convergent in the interior of G, should have an analytical limiting function. Necessary it is that the convergence of the sequence be quasi-uniform in every closed region within G, because of the continuity of f_n and f. If, conversely, this quasi-uniform convergence is given, then it follows that f is continuous within G, but not that f is analytical, as an example of MONTEL[2]) shows. In the following a necessary and sufficient condition will be deduced among other things.

I.

We consider a sequence of real functions $f_1(x), f_2(x), \ldots$, each of which is continuous for $0 \leqq x \leqq a$; the sequence converges for $0 < x \leqq a$. Let $f(x)$ be the limiting function.

1. The function $\Phi_n(x,y) = f_n(x) - f_n(y)$ is definite in the rectangle $0 \leqq x \leqq a$, $0 \leqq y \leqq a$ and is there continuous. The sequence of functions $\Phi_1(x,y)$, $\Phi_2(x,y), \ldots$ converges at every point of the rectangle where not simultaneously $x > 0$ and $y = 0$, or $x = 0$ and $y > 0$. The limiting function is $\Phi(x,y) = f(x) - f(y)$ at every point where $x > 0$ and $y > 0$ and vanishes at $x = 0$, $y = 0$, since there every $\Phi_n = 0$. Let it be given that $f(x)$ converges to a finite limit as $x \to 0$. We consider a set of points $P_1(x_1, y_1)$, $P_2(x_2, y_2), \ldots$ where $0 < x_k \leqq a$, $0 < y_k \leqq a$, $\lim_{k=\infty} x_k = 0$, $\lim_{k=\infty} y_k = 0$; two arbitrary positive numbers may be given, ε and N. Let N_0 denote an arbitrary integer $> N$.

We have $\Phi_{N_0}(0,0) = 0$. In consequence of the continuity of Φ_{N_0} a number δ_1 can be assigned such that at every point P_k, where

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[1]) P. MONTEL. Bul. des Sc. math., série 2, tome 30, part 1 (vol. 41, 1906, p. 191).
[2]) P. MONTEL, l. c. p. 190.
" Thèse p. 98.

$x_k < \delta_1$, and $y_k < \delta_1$ the inequality $|\Phi_{N_0}(x_k, y_k)| < \frac{\varepsilon}{2}$ is satisfied.

Since $f(x)$ tends to a finite limit as $x \to 0$ there is a number δ_2 such that for $x_k < \delta_2$ and $y_k < \delta_2$ the inequality $|\Phi(x_k, y_k)| < \frac{\varepsilon}{2}$ is satisfied.

If $\delta < \delta_1$ and at the same time $< \delta_2$ then

$$|\Phi(x_k, y_k) - \Phi_{N_0}(x_k, y_k)| < \varepsilon \quad \text{for} \quad x_k < \delta \text{ and } y_k < \delta.$$

There now exist only a finite number of indices for which $x_k \geq \delta$, $y_k \geq \delta$. At each of the points P_k corresponding there to an index $N_k > N$ can be found such that

$$|\Phi(x_k, y_k) - \Phi_{N_k}(x_k, y_k)| < \varepsilon.$$

At the points of the considered set the sequence of functions Φ_n therefore converges quasi-uniformly.

2. Conversely, let it be given that at the points of every set $P_1(x_1, y_1)$, $P_2(x_2, y_2), \ldots$ where $0 < x_k \leq a$, $0 < y_k \leq a$, $\lim_{k=\infty} x_k = 0$, $\lim_{k=\infty} y_k = 0$, this sequence of functions converges quasi-uniformly.

Let M denote the maximum of $f(x)$ at 0, m the minimum. It is possible to construct a set of points $x_1, x_2, \ldots$ where $x_k > 0$ and $\lim_{k=\infty} x_k = 0$, such that $\lim_{k=\infty} f(x_k) = M$; similarly we can construct a set $y_1, y_2, \ldots$, where $y_k > 0$ and $\lim_{k=\infty} y_k = 0$, such that $\lim_{k=\infty} f(y_k) = m$.

If we now consider the set of points

$$P_1(x_1, y_1), \quad P_2(x_2, y_2), \ldots, \quad \text{then} \quad \lim_{k=\infty} \Phi(x_k, y_k) = M - m, \quad (1)$$

unless $m = +\infty$ or $M = -\infty$. Now it is possible, by hypothesis, to choose at each of the points P_k from a finite number of indices $1, 2, \ldots n(\varepsilon)$ an index n_k such that

$$|\Phi(x_k, y_k) - \Phi_{n_k}(x_k, y_k)| < \frac{\varepsilon}{2}. \quad \ldots \ldots \quad (2)$$

where ε denotes a given arbitrary positive number. In consequence of the continuity of each of the functions $\Phi_{n_k}(x, y)$ and since the number of indices here considered is finite, a number δ exists such that at every point P_k, where $x_k < \delta$ and $y_k < \delta$ the relation

$$|\Phi_{n_k}(0, 0) - \Phi_{n_k}(x_k, y^k)| < \frac{\varepsilon}{2},$$

is satisfied.

Hence since

$$\Phi_{n_k}(0\ 0) = 0, \qquad \text{we have } |\Phi_{n_k}(x_k, y_k)| < \frac{\varepsilon}{2}. \quad \ldots \ldots \quad (3)$$

From (2) and (3) it follows that $|\Phi(x_k, y_k)| < \varepsilon$ for $x_k < \delta$, $y_k < \delta$ and from this, since $\lim_{k=\infty} x_k = 0$ and $\lim_{k=\infty} y_k = 0$,

$$\lim_{k=\infty} \Phi(x_k, y_k) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

From (1) and (4) it follows that $M = m$, hence $\lim_{x \to 0} f(x)$ exists and is finite, unless $m = +\infty$ or $M = -\infty$.

If however $m = +\infty$, then at the points of every set $x_1, x_2, \ldots$ where $x_k > 0$ and $\lim_{k=\infty} x_k = 0$, the $\lim_{k=\infty} f(x_k)$ is $+\infty$.

To every x_k of such a set a $y_k > 0$ can be found such that $f(y_k) > f(x_k) + k$ and $\lim_{k=\infty} y_k = 0$. Then $\lim_{k=\infty} \Phi(x_k, y_k) = -\infty$, which is incompatible with the formula (4). Similarly it appears that M cannot possibly be $-\infty$. Collecting our results we have the following theorem:

If a sequence of functions $f_n(x)$ be given each of which is continuous for $0 \leqq x \leqq a$, and if $\lim_{n=\infty} f_n(x) = f(x)$ for $0 < x \leqq a$, then a necessary and sufficient condition that $f(x)$ should tend to a finite limit at 0 is: the quasi-uniform convergence of the sequence of functions $\Phi_n(x, y) = f_n(x) - f_n(y)$ at the points of every set $(x_1, y_1), (x_2, y_2), \ldots$ where $0 < x_k \leqq a$, $0 < y_k \leqq a$ and $\lim_{k=\infty} x_k = 0$, $\lim_{k=\infty} y_k = 0$.

II.

3. Let $f_1(x), f_2(x), \ldots$ be continuous for $0 \leqq x \leqq a$ with finite derivatives $f_n'(0)$ at 0. Let the series be convergent for $0 < x \leqq a$. The functions $f_n^*(x) = \dfrac{f_n(x) - f_n(0)}{x}$ for $x > 0$ and $f_n'(0)$ for $x = 0$ are in consequence of the suppositions continuous at $0 \leqq x \leqq a$; at $0 \leqq x \leqq a$ the sequence of functions $f_n^*(x)$ converges to the limiting function $f^*(x) = \dfrac{f(x) - f(0)}{x}$, where $f(x) = \lim_{n=\infty} f_n(x)$.

If $f(x)$ has a finite derivative at 0, then $f^*(x)$ tends to a finite limit for $x \to 0$, and *vice versa*.

Hence, by applying the theorem demonstrated in I, we find, if we take into consideration that

$$f_n^*(x) - f_n^*(y) = \frac{y f_n(x) - x f_n(y) + (x - y) f_n(0)}{xy}:$$

If a sequence of functions $f_n(x)$ *is given each of which is continuous for* $0 \leqq x \leqq a$ *and has a finite derivative* $f_n'(0)$ *at* 0, *and if, for* $0 \leqq x \leqq a$, $\lim_{n=\infty} f_n(x) = f(x)$, *then a necessary and sufficient condition that* $f(x)$ *should have a finite derivative at* 0 *is: the quasi-uniform convergence of the sequence of functions* $\Psi_n(x,y) = \dfrac{y f_n(x) - x f_n(y) + (x-y) f_n(0)}{xy}$ *at the points of every set* (x_1, y_1), $(x_2, y_2), \ldots$ *where* $0 < x_k \leqq a$, $0 < y_k \leqq a$, *and* $\lim_{k=\infty} x_k = 0$, $\lim_{k=\infty} y_k = 0$.

4. We observe that in this theorem nothing is supposed about the convergence or divergence of the sequence $f_n'(0)$. Let this sequence be convergent. Then the sequence of continuous functions $f_n^*(x)$ is convergent for $0 \leqq x \leqq a$ and converges to $f^*(x)$ for $x > 0$, to $\lambda = \lim_{n=\infty} f_n'(0)$ for $x = 0$.

If $f'(0) = \lim_{n=\infty} f_n'(0)$ then

$$\lim_{x=0} f^*(x) = \lambda \quad . \quad . \quad . \quad ; \quad . \quad . \quad . \quad . \quad (5)$$

We now consider a set of points $x_1, x_2, \ldots$, where $\lim_{k=\infty} x_k = 0$. By ε, N we denote two given positive numbers. Then there exists an index $N_0 > N$ such that

$$|f_{N_0}'(0) - \lambda| < \varepsilon \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Since $f_{N_0}'(0) = f_{N_0}^*(0)$ and $f_{N_0}^*(x)$ is continuous and because of (5), there exists a certain number k_0 such that for every $k > k_0$

$$|f_{N_0}^*(x_k) - f^*(x_k)| < \varepsilon.$$

Since at each of the finite number of points where $k \leqq k_0$ an index $N_k > N$ can be found such that

$$|f_{N_k}^*(x_k) - f^*(x_k)| < \varepsilon,$$

it appears that the sequence of functions $f_n^*(x)$ converges quasi-uniformly at the points of the considered set.

If this sequence of functions converges quasi-uniformly at the points of every set of the above-considered kind, then it results from an argument as held in § 2 that

$$f'(0) = \lim_{n=\infty} f_n'(0).$$

We have therefore the following theorem:

If a series of functions $f_n(x)$ *be given, each of which is continuous*

for $0 \leqq x \leqq a$ *and has a finite derivative* $f_n'(0)$ *at* 0, *and if* $f_n(x)$ *for* $0 \leqq x \leqq a$ *converges to* $f(x)$ *and the sequence* $f_n'(0)$ *is convergent, then a necessary and sufficient condition that it should be permitted to "differentiate term-wise at* 0" *i.e. that* $\lim_{n=\infty} f_n'(0)$ *should be* $f'(0)$, *is: the quasi-uniform convergence of the sequence of functions*

$$f_n^*(x) = \frac{f(x) - f_n(0)}{x}$$

at the points of every set $x_1, x_2, \ldots$ *where* $\lim_{k=\infty} x_k = 0$.

The function $f_n(x) = \frac{x}{1+n^2x^2}$ converges everywhere to zero, hence $f'(0) = 0$,

$$f_n^*(x) = \frac{1}{1+n^2x^2}, \quad f_n'(0) = 1, \quad \psi_n(x,y) = \frac{1}{1+n^2x^2} - \frac{1}{1+n^2y^2}.$$

$\psi_n(x, y)$ everywhere converges to a continuous function, *i. e.* quasi-uniformly in every finite region so that the criterion given in § 3 is satisfied. Accordingly $f'(0)$ is finite. However, if we choose $\varepsilon < 1/2$ and the number of indices is finite, then for every x which is small enough we have for each of these indices $f_n^*(x) > 1/2$, whilst for $x \neq 0$ we have $\lim_{n=\infty} f_n^*(x) = 0$. The criterion, enunciated in this § is therefore not satisfied. Accordingly we have

$$\lim_{\infty} f_n(0) = 1 \neq f'(0).$$

The foregoing theorems can be immediately extended to the complex plane and are capable of analogous proofs.

III.

5. Let there be given a series of functions $f_1(z), f_2(z), \ldots$, each of which is analytical in the interior of the circle $|z| < a$, and convergent within this region. We shall now demonstrate the following theorem:

A necessary and sufficient condition that the limiting function f *should be analytical within* $|z| < a$ *is the quasi-uniform convergence of the sequence of functions*

$$\psi_n(x, y, z) = \frac{(y-z) f_n(x) + (x-y) f_n(z) + (z-x) f_n(y)}{(x-z)(y-z)} \quad . \quad (1)$$

at the points of every closed and limited set $V(x, y, z)$ *having no point in common with the sets* $(x = z,\ y \neq z)$ *and* $(x \neq z,\ y = z)$.

For $\psi_n(x, y, z)$ may be written $\Phi_n(x, z) - \Phi_n(y, z)$, where

$$\Phi_n(u, v) = \frac{f_n(u) - f_n(v)}{u - v} \quad \ldots\ldots\ldots \quad (2)$$

Since the functions f_n are for $|z| < a$ analytical $\psi_n(x, y, z)$ is an analytical function of x, y and z in the region $|x| < a$, $|y| < a$, $|z| < a$, if only for $u = v$ we put $\Phi_n(u, v) = \Phi_n'(u)$, which has been tacitly assumed in the above.

6. Let it be given that f is analytical. Let (x_0, y_0, z_0) denote a point of V, and $x_0 \neq z_0$, then $y_0 \neq z_0$.

We then have

$$\lim_{n=\infty} \psi_n(x_0, y_0, z_0) = \frac{(y_0 - z_0) f(x_0) + (x_0 - y_0) f(z_0) + (z_0 - x_0) f(z_0)}{(x_0 - z_0)(y_0 - z_0)} = \psi(x_0, y_0, z_0).$$

In a sufficiently small neighbourhood Ω' of $(x_0\, y_0\, z_0)$ we have also

$$\lim_{n=\infty} \psi_n(x, y, z) = \frac{(y - z) f(x) + (x - y) f(z) + (z - x) f(y)}{(x - z)(y - z)} = \psi(x, y, z).$$

The being analytical of f involves the same of $\psi(x, y, z)$ within Ω' and in particular the continuity of $\psi(x, y, z)$. If ε and N are given then an index $N_0 > N$ exists such that

$$|\psi_{N_0}(x_0, y_0, z_0) - \psi(x_0, y_0, z_0)| < \varepsilon.$$

From the continuity of ψ_{N_0} and ψ within Ω' it follows that in Ω' is contained a region Ω in which everywhere

$$|\psi_{N_0}(x, y, z) - \psi(x, y, z)| < \varepsilon \quad \ldots\ldots\ldots \quad (3)$$

If $x_0 = y_0 = z_0$ then $\psi_n(x_0, y_0, z_0) = 0$ for every n. Also ψ_n is continuous in a neighbourhood Ω' of (x_0, y_0, z_0) and moreover we have $\lim_{\substack{x=x_0 \\ y=y_0 \\ z=z_0}} \psi(x, y, z) = 0$, so that here too the inequality (3) is satisfied in a neighbourhood Ω of (x_0, y_0, z_0).

Every point of V therefore lies in a neighbourhood Ω such that throughout ΩV by one and the same index N_0 the relation (3) is satisfied. Since V is limited and closed V can be covered by a finite number of these neighbourhoods [1]), whence it follows that ψ_n converges quasi-uniformly at the points of V.

7. Conversely, let be given the quasi-uniform convergence of ψ_n at the points of every V. Let z_0 denote an arbitrary point where

[1]) E. Borel. Leçons sur les Fonctions monogènes, p. 11.

$|z_0| < a$. For V we now choose an arbitrary set of points $P_0(z_0, z_0, z_0)$, $P_1(x_1, y_1, z_0)$, $P_2(x_2, y_2, z_0), \ldots$ where $x_k \neq z_0$, $y_k \neq z_0$, $\lim_{k=\infty} x_k = z_0$, $\lim_{k=\infty} y_k = z_0$.

Ex hypothesi it is possible at every point P_k of V to choose n_k such that

$$|\psi_{n_k}(x_k, y_k, z_0) - \psi(x_k, y_k, z_0)| < \frac{\varepsilon}{2} \quad \ldots \quad (4)$$

Since ψ_n is continuous and the number of indices n_k is finite a positive number δ exists such that for $x_k - z_0 < \delta$ and $y_k - z_0 < \delta$ we have

$$|\psi_{n_k}(x_k, y_k, z_0) - \psi_{n_k}(z_0, z_0, z_0)| < \frac{\varepsilon}{2} \quad \ldots \quad (5)$$

Since $\psi_{n_k}(z_0, z_0, z_0) = 0$, it follows from (4) and (5) that

$$|\psi(x_k, y_k, z_0)| < \varepsilon \text{ for } |x_k - z_0| < \delta \text{ and } |y_k - z_0| < \delta,$$

$$\text{i. e. } \left| \frac{f(x_k) - f(z_0)}{x_k - z_0} - \frac{f(y_k) - f(z_0)}{y_k - z_0} \right| < \varepsilon.$$

It results from this, that $\frac{f(x_k) - f(z_0)}{x_k - z_0}$ tends to a finite limit as $x_k \neq z_0$ coincides successively with the points of an arbitrary set of which z_0 is the sole limiting point, and from this again that a finite $\lim_{x \to z_0} \frac{f(x) - f(z_0)}{x - z_0}$ exists, hence $f(z)$ has a finite derivative at z_0. Since this holds good for any $z < a$ it follows from a theorem enunciated by GOURSAT that $f(z)$ is analytical within $|z| < a$.

It may be stated that here only a part of the supposition has been made use of.

8. *If the sequence of functions $f_1(z), f_2(z), \ldots$, each of which is analytical for $z < a$, converges at a point z_0 internal to this region and the function $f_n^*(x, y) = \frac{f_n(x) - f_n(y)}{x - y}$ converges quasi-uniformly at the points of every closed set internal to $|x| < a, |y| < a$, then $f_n(z)$ converges for every $z < a$, the limiting function f is analytical and everywhere there is*

$$f'(z) = \lim_{n=\infty} f_n'(z).$$

That $f_n(z)$ converges everywhere is involved in the convergence

of $\frac{f_n(z)-f_n(z_0)}{z-z_0}$. For $x=y=z$ $f_n^*(x,y)$ coincides with $f_n'(z)$, hence $f_n'(z)$ converges also. If C denotes a circle situated within $|z|<a$, with centre z_0, then $f_n^*(z_0,z)$ converges quasi-uniformly within C. This function is continuous in C and for $z=z_0$ coincides with $f_n'(z)$. In consequence of the quasi-uniform convergence the limiting function is also continuous within C. This limiting function is $\frac{f(z)-f(z_0)}{z-z_0}$ for $z \neq z_0$.

Hence

$$\lim_{z=z_0} \frac{f(z)-f(z_0)}{z-z_0} = \lim_{n=\infty} f_n'(z_0),$$

whence it follows that $f'(z_0)$ does exist and $f'(z_0) = \lim_{n=\infty} f_n'(z_0)$. Since z_0 may be chosen arbitrarily within $|z|<a$ the theorem is hereby established.

9. *If the sequence of functions $f_1(z), f_2(z), \ldots$, each of which is analytical for $|z|<a$, in this region converges everywhere to an analytical function $f(z)$ and if besides for any z the following relation holds*

$$f'(z) = \lim_{n=\infty} f_n'(z),$$

then the function $f_n^*(x,y) = \frac{f_n(x)-f_n(y)}{x-y}$ converges quasi-uniformly at the points of every closed set internal to $|x|<a$, $|y|<a$.

In consequence of f_n being convergent $f_n^*(x,y)$ converges to $\frac{f(x)-f(y)}{x-y}$ for $x \neq y$ and to $f'(z)$ for $x=y=z$, since there $f_n^*(x,y) = f_n'(z)$. The function $f_n^*(x,y)$, which is everywhere continuous therefore converges to a function which, since f is analytical, is also everywhere continuous. This involves the quasi-uniform convergence of $f_n^*(x,y) = f_n'(z)$ at the points of every closed set within $|x|<a$, $|y|<a$.

10. The theorems of § 8 and 9 may be resumed concisely as follows:

A necessary and sufficient condition that it should be permitted to differentiate termwise a convergent series of analytical functions is the quasi-uniform convergence of the series $\Sigma f_n^*(x,y)$.

11. *If within $|z|<a$ a convergent series of analytical functions*

$f_n(z)$ is given, such that the limiting function $f(z)$ has a finite derivative at 0, and if moreover the series $f_1^*(0,z)+f_2^*(0,z)+\ldots$ converges quasi-uniformly at $0<|z|\leqq b<a$, then it is possible to assemble the terms of the series in groups in such a way that the new series may be differentiated termwise at 0.

If we put $\sum_1^n f_n(z)=S_n(z)$ and $\sum_1^n f_n^*(0,z)=S_n^*(0,z)$, then $S_n^*(0,z)$ is continuous for $0\leqq|z|\leqq b$ and $S_n^*(0,0)=S_n'(0)$.

Besides, for $|z|>0$ we have

$$\lim_{n=\infty} S_n^*(0,z)=\frac{f(z)-f(0)}{z}=f^*(0,z).$$

By hypothesis this function tends to a finite limit $f'(0)$ as $z\to 0$.

Now let an arbitrary number $\varepsilon_1>0$ be given. We construct a set of points $z_1, z_2, \ldots$ where $\lim_{k=\infty} z_k=0$. From a limited number of indices an index can be chosen at each of these points such that

$$|S_{n_k}^*(0,z_k)-f^*(0,z_k)|<\varepsilon_1$$

Hence an infinite number of points z_k exists where one and the same index can be used which we denote by n_1. As z_k tends to zero $S_{n_1}^*(0,z_k)$ tends to $S'_{n_1}(0)$ and $f^*(0,z_k)$ to $f'(0)$. Thence it follows that $|S'_{n_1}(0)-f'(0)|\leqq\varepsilon_1$. Let $\varepsilon_1,\varepsilon_2,\ldots$ be a decreasing sequence of positive numbers, having zero as limit. It is again possible to choose for every z_k out of a limited number of indices an index $n_k>n_1$ such that $|S_{n_k}^*(0,z_k)-f^*(0,z_k)|<\varepsilon_2$, whence may be concluded, as before, to the existence of an index $n_2>n_1$ such that $|S'_{n_2}(0)-f'(0)|\leqq\varepsilon_2$.

Thus pursuing we find that there is a partial series of functions: $S'_{n_1}(z)$, $S'_{n_2}(z),\ldots$, such that $\lim_{p=\infty} S'_{n_p}(0)=f'(0)$, which establishes the theorem [1]).

12. The theorem of the foregoing § may be reversed as follows:

If within $|z|<a$ a convergent sequence of analytical functions $f_n(z)$ is given such that the limiting function $f(z)$ is continuous for $z\neq 0$ and has a finite derivative $f'(0)$ at 0, and if moreover a partial sequence $f_{n_1}(z), f_{n_2}(z),\ldots$ can be found where $\lim_{p=\infty} f'_{n_p}(0)=f'(0)$, then the sequence of functions $f_n^*(0,z)$ converges quasi-uniformly for $0<|z|\leqq b$, where b is an arbitrary number $<a$.

We denote by ε and N two given positive numbers. Then a number $n_p>N$ can be determined such that

$$|f'(0)-f'_{n_p}(0)|<\varepsilon.$$

[1]) Verh. Ac. v. W., vol. 27 (1919), p. 1102.

Since $\lim_{z=0} f^*(0, z) = f'(0)$ and $\lim_{z=0} f^*_{n_p}(0, z) = f'_{n_p}(0)$, it is possible to determine a number δ such that

$$| f^*(0, z) - f^*_{n_p}(0, z) | < \varepsilon \quad \text{for} \quad | z | < \delta.$$

Since at the points of the annular region $\delta \leqq |z| \leqq b$ the function $f^*_n(0, z)$ converges to the continuous function $f^*(0, z)$, this convergence is quasi-uniform there *i. e.*: from a finite number of indices $> N$ at every point z of the annular region an index n_z may be chosen which satisfies

$$| f^*(0, z) - f^*_{n_z}(0, z) | < \varepsilon.$$

At every point $0 < |z| \leqq b$ therefore from a finite number of indices $> N$ such a choice can be made, which establishes the theorem.

13. Some of the results here obtained we shall apply to MONTEL's example, which was cited in the introduction and is reproduced below:

The function $\varphi_n(z) = n^2 z e^{-n^2 z^2}$ tends to zero as $n \to \infty$ for all real values of z. Now consider the three rectangles $I_n\left(-n \leqq x \leqq n, -\frac{1}{2n} \leqq y \leqq \frac{1}{2n}\right)$, $II_n\left(-n \leqq x \leqq n, \frac{1}{n} \leqq y \leqq n\right)$ and $III_n\left(-n \leqq x \leqq n, -n \leqq y \leqq -\frac{1}{n}\right)$.

There exists a polynomium $P_n(z)$ which within I_n differs less than $\frac{1}{4n}$ from $\varphi_n(z)$ and within II_n and III_n less than $\frac{1}{n}$ from zero. Evidently the function $P_n(z)$ tends to zero for $n \to \infty$ throughout the whole plane. According to the theorem established in § 5 the function $\psi_n(x, y, z)$ must converge quasi-uniformly at the points of every set V: $P_1(x_1, y_1, 0)$, $P_2(x_2, y_2, 0), \ldots$ where $x_k \neq 0$, $y_k \neq 0$, $\lim_{k=\infty} x_k = 0$ and $\lim_{k=\infty} y_k = 0$. That this is really the case may be proved as follows: Choose an arbitrary pair of numbers ε, N. Let N_0 be $> N$ and at the same time $> \frac{1}{2\varepsilon}$. From a certain index k_0 onwards x_k and y_k are both internal to I_{N_0}, so that

$$\left| \psi_{N_0}(x_k, y_k, 0) - N_0^2 (e^{-N_0^2 x_k^2} - e^{-N_0^2 y_k^2}) \right| < \frac{1}{4N_0} < \frac{\varepsilon}{2}$$

Hence a number δ exists such that for $| x_k | \leqq \delta$ and $| y_k | \leqq \delta$ we have

$$| \psi_{N_0}(x_k, y_k, 0) | < \varepsilon$$

This, we remind, is the case from a certain index k_1 onwards.

Besides, a number $N_1 > N$ can be found satisfying $N_1 > \frac{4M}{m^2}$,

where M denotes the maximum of x_k and y_k, m the minimum for $k = 1, 2, \ldots k_1 - 1$,) and such that at the same time $\frac{1}{N_1} < |x_k| < N_1$ and $\frac{1}{N_1} < |y_k| < N_1$ for $k = 1, 2, \ldots k_1 - 1$. The points $P_1, P_2, \ldots P_{k_1-1}$ then are contained in $II_{N_1} + III_{N_1}$, and we have

$$\left| \psi_{N_1}(x_k, y_k, 0) \right| < \frac{4M}{N_1 m^2} < \varepsilon.$$

Since now at every point of the plane $\lim \psi_n = 0$, at each point of V one of the two indices N_0 and N_1 satisfies the relation $|\psi_n - \lim_{n=\infty} \psi_n| < \varepsilon$, which proves the quasi-uniform convergence. Besides at 0 the derivative of the limiting function is zero, and $\lim_{n=\infty} P_n'(0) = +\infty$ [1]).

It follows from § 8 that the function $P_n^*(x, y)$ cannot be quasi-uniformly convergent at the points of every set V. For V we choose: $x = 0$, $0 \leqq y \leqq a$. Since the continuous function $P_n^*(x, y)$ for $y \neq 0$ converges to zero and for $y = 0$ to $+\infty$, the convergence is not quasi-uniform.

14. *If in a region G a series of analytical functions converges to an analytical function and if, besides, the series of derived functions converges quasi-uniformly at the points of every closed set contained in G, then it is permitted to differentiate the series termwise everywhere within G.*

In the first place we observe that every region within G contains another region where the convergence is uniform [2]), so that the termwise differentiation is permitted in this last region.

Since $f_n'(z)$ converges to a function which is continuous within G and coincides on an everywhere dense set of points with $f'(z)$, we have everywhere $f'(z) = \lim_{n=\infty} f_n'(z)$.

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[1]) For any n, indeed,

$$|P_n'(0) - \varphi_n'|(0) = \frac{1}{2\pi} \left| \int_{I_n} \frac{\{P_n(t) - \varphi_n(t)\}\, dt}{t^2} \right| < \frac{4n^2 + 2}{2\pi} < \frac{2n^2 + 1}{3}$$

Since $\varphi_n'(0) = n^2$, we have $|P_n(0)| > \frac{n^2 - 1}{3}$.

[2]) P. Montel. Thése, p. 83.

Physiology. — *"On Spray-Electricity and Waterfall-Electricity"*. By Prof. H. ZWAARDEMAKER and Dr. F. HOGEWIND.

(Communicated in the meeting of October 25, 1919).

The generation of Spray-electricity and that of Waterfall-electricity are no doubt cognate processes; still they are by no means identical. It may perhaps be useful, therefore, that we should here enlarge upon their congruency and their difference.

Spray-electricity is generated when the air causes waterdrops to break up and diffuse; waterfall-electricity is evolved when existing waterdrops strike against a boundary plane of air-liquid or air-solid substance. This induces electrification of the spray-nebula at the very spot where the cloud arises, whereas the electrical charge of the waterfall does not take its beginning before the water reaches the bottom. In either case small and large drops are formed with opposite charges. Both with spray-electricity and with waterfall-electricity the surrounding air is laden to a large distance with those diminutive droplets, driven off in all directions.

With spraying the large drops follow their primitive course till they strike on some impediment or other. These drops have become electrified long before they encounter this impediment. In the case of waterfall-electricity, however, large drops as well as small ones form at the very moment when the electric charge begins, i.e. the moment when the jet collides with the impediment.

In either case the conditions of

"pressure"
and "temperature"

largely reinforce the electrical effect. An overpressure of two atmospheres yields notably more electricity than one atmosphere. To obtain a considerable reinforcement of spray-electricity it is only necessary to store up the nebula in a space, whose temperature is 10° higher. Likewise waterfall-electricity will be considerably increased by heating the reservoir from which the waterfall proceeds. The presence of an electric field will augment either in a marked degree.

The distance at which the disc, doing duty for an impediment, is arranged, has an influence upon either.

In the case of spray-electricity there is an optimal and a critical distance. According to LENARD [1]) waterfall-electricity augments with the distance of the disc.

However, the liquid medium and the small chemical additions are of consequence. In most experiments water was the medium, but paraffinum liquidum will also generate spray-electricity, while waterfall-electricity can be obtained also with mercury. We have chiefly noted the influence of small additions of known chemical nature.

Spray-electricity is markedly raised by the addition of substances that lower the surface-tension and are moreover volatile (odorous matter, antipyretica, narcotica, alkaloids). It may thereby rise to an amount which waterfall-electricity cannot approach by far.

Perfectly pure water does not yield spray-electricity that is distinguishable with an ordinary electroscope. [2])

Traces of odorous matter are capable of rendering it excessive.

Spray-electricity, therefore is a means to detect the presence of small amounts of odorous matter otherwise than by the sense of smell.

Pure water of itself generates waterfall-electricity, as the water becomes positively electric at the moment of its collision.

Minute additions of odorous matter, gustatory substances, colloidal substances, modify the charge of the water considerably, now in a positive, now in a negative sense.

LENARD was the first who studied this problem in 1892, and immediately put forward an interpretation [3]), which he altered a little and extended in a subsequent publication. [4])

In our experiments we made use of LENARD's apparatus, with slight alterations.

A strong metal cylinder of $\pm 2\frac{1}{2}$ liter capacity, with the lid attached to it hermetically by three screws, is provided at the bottom with a metal pipe with a tap. The pipe terminates in a glass tube with a fine outlet (1 mm. in diameter). In the lid there are two apertures, one of which is connected with the supply-pipe of the compressor with a gas-chamber of 2 m.^{3} capacity, while the other which is closed with a screw-stopper, subserves the filling of the cylinder.

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[1]) P. LENARD: "Ueber die Elektricität der Wasserfälle". Wied. Ann. 46 p. 584. 1892

[2]) Unless in the presence of an electric field.

[3]) P. LENARD: "Ueber die Elektricität der Wasserfälle". Wied. Ann. 46—1892.

[4]) Id. id. "Ueber Wasserfallelektricität". Ann. der Physik. Bd. 47—1915.

Inside the supply-pipe of the gas-chamber an amber tube of ± $2^1/_2$ cm. is fitted for isolation, besides two taps: the one close to the cylinder, the other near the main-pipe leading to the gas-chamber.

The whole cylinder is suspended in a trivet from which it is insulated by amber pins.

The outlet is placed at about 1 m. over a large glass receiving-reservoir, in which a zinc plate rests on two wooden blocks. This tank is connected by a conducting-wire with the metal cylinder.

In its turn the reservoir is isolated from the environment by a paraffin-plate supported on four amber feet.

The whole apparatus is connected by a conducting-wire to an earthed electroscope.

Thus the cylinder, the receiving-reservoir and the electroscope are connected inter se by an electric circuit; they are at the same time insulated from the environment.

The pressure in the air-pump and the gas-chamber, registered by a manometer, is brought up to two atmospheres, the cylinder is filled with 1 Liter of the liquid to be examined, and the two taps in the supply-pipe are opened, so that the liquid in the cylinder is subjected to a pressure of 2 atmospheres. Now when the lower tap is turned on, the fluid flows under a high pressure out of the glass tube and strikes at an angle of 90° against the zinc plate below it. This produces positive or negative electricity according to the nature of the liquid and causes a deflection of the electroscope.

The deflection, registered by the electroscope after 1 minute's perfusion, is taken as the index for waterfall-electricity.

The entire apparatus being of a rather large capacity the electroscope takes some time before deflecting, which does not occur before the whole capacity is electrified. This takes more time with some liquids than with others.

For this reason the stopwatch is not put in operation before the electroscope begins to deflect and the liquid then continues flowing for a full minute after this.

During the experiment the room is well aired, because the air in the room is also charged and that in a sense opposite to the charge of the liquid. It is obvious that this would greatly interfere with the electrification of the liquid in the subsequent experiments.

Moreover, the receiving reservoir is covered with a close-mesh iron gauze, provided with a circular opening in the middle, through which the jet passes. This gauze serves to keep back the migrating droplets and possible foam, and to allow the extremely hazy nebula to spread in the surrounding air.

Our standard-liquid was pure tapwater, which yielded a mean deflection of 50 scale-marks. The Utrecht tapwater is very pure and contains few salts. In addition the temperature is pretty constant, which is of great importance, since temperature very much influences water-electricity.

A control-test with water was inserted between every set of two experiments, in order to ascertain the accuracy of the apparatus.

Furthermore the cylinder was washed out with tapwater after each experiment to remove small quanta of lingering electrifying substances, which might render the results of the following experiments less reliable.

In the case of water-electricity we found that odorous substances did not all act in the same way. Most of them reinforced the positive charge of the water; others hardly modified it or did not do so at all; a few again even weakened it so as to excite a negative charge. All this occurred seemingly without any special method. It is true, stronger concentrations (which are insignificant with the almost always slightly soluble odorous substances) generally give a greater increase or decrease. Besides, in the homologous series we found an augmentation of the deflection according as we passed from the lower to the higher terms.

For the present it seems utterly impossible to draw a hard-and-fast line, separating the reinforcing from the weakening odorous substances.

That waterfall-electricity is not identical with spray-electricity, may appear e. g. from the behaviour of indol, which markedly increases the charge in the former, but is almost inoperative with the second.

Conversely thymol e. g. gives a strong nebula-charge and hardly any waterfall-electricity.

Another instance is that of *fresh*-distilled water, which very distinctly intensified waterfall-electricity, whereas it remains inactive in spraying.

We subjoin a list of some odorous substances with their charges in the numerator of the fraction and the deflection of tapwater in the denominator, as we noted them down directly after the reading. The sign of the charge is also given:

Phenol $^1/_{1000}$ N. sol. : $+ \, ^{100}/_{80}$
Cressol $^1/_{1000}$ N. sol. : $+ \, ^{110}/_{80}$
Xylenol $^1/_{1000}$ N. sol. : $+ \, ^{120}/_{80}$

Amylacetate $^1/_{1000}$ N. sol.:	$+ ^{130}/_{35}$	1 % aethylalcohol.	$+ ^{100}/_{50}$
Thymol sol. (sat.).	$+ ^{40}/_{35}$	5 % „	$+ ^{90}/_{50}$
Toluol (dil. sol.)	$+ ^{100}/_{50}$	10 % „	$+ ^{20}/_{50}$
Artificial Moschus (dil.)	$+ ^{65}/_{50}$	20 % „	$- ^{20}/_{50}$
1 % Amylalcohol	$+ ^{100}/_{50}$		
Indol. sol. (dil.)	$+ ^{100}/_{50}$	Bornylacetate (sat.)	$+ ^{40}/_{50}$
Encalyptol (sat.)	$+ ^{120}/_{50}$	Camphor sol. („)	$- ^{30}/_{50}$
Safrol (sat.)	$+ ^{100}/_{50}$	Capronic acid sol.	$+ ^{10}/_{50}$
Citrol (sat.)	$+ ^{90}/_{50}$	Acetic acid $^1/_8$ % sol.	$+ ^{30}/_{50}$
		„ 1 % „	$- ^{10}/_{50}$

From this it appears that most odorous substances increase the positive charge of the water. Only bornylacetate and camphor two substances closely allied as to smell, lower, resp. reverse, the charge.

The odorous substances that belong to the acids, lessen the charge of the water, just as all other acids do.

Finally very strong concentrations of odorous substances, such as we can procure with aethylalcohol, will lessen the charge or will even produce a negative charge, just as is the case with spray-electricity.

When passing on to *gustatory substances* we found, that on the one hand all sweet substances are more or less reinforcing, though the concentration must be stronger than in the case of odorous substances, and that, on the other hand, all salts and acids lessen water-electricity, while such bitter substances as belong to the electrolytes, also lessened it.

Bitter substances, however, that must be classed under the colloidal substances again raised the positive charge of water, just as all the other colloidal substances examined.

Sweet substances.

Saccharose 1 % :	$+ ^{70}/_{35}$
Laevulose 1 % :	$+ ^{100}/_{50}$
Glucose $^1/_4$ % :	$+ ^{85}/_{50}$
„ 1 % :	$+ ^{100}/_{50}$
Sacch. lactis 1 % :	$+ ^{120}/_{50}$
Glucocoll $^1/_4$ % :	$+ ^{120}/_{50}$
Glycerin 1 % :	$+ ^{100}/_{50}$

Bitter substances.

Chinin pur (sat. sol.).	$- ^{30}/_{50}$
Bisulj. chin. 1 %	$- ^{30}/_{65}$
Chloret. magnes 1 %	$- ^{15}/_{50}$
Sulfas natricus 1 %	$- ^{15}/_{50}$
Chloret. plumb. $^1/_2$ %	$- ^{5}/_{50}$

Colloidal Bitter substances.

Fel Tauri $^1/_{1000}$:	$+ ^{100}/_{50}$
Extr. quassiae sicc. $^1/_{800}$:	$+ ^{100}/_{75}$
Glucochol. acid natr. $^1/_{1000}$:	$+ ^{80}/_{50}$.

Salts.

Sulfas natric.	1 % :	$-\frac{15}{50}$
Sulf. kalic.	1 % :	$-\frac{25}{50}$
Sulf. ammon.	1 % :	$-\frac{30}{50}$
Chloret magn.	1 % :	$-\frac{18}{50}$
„ natr.	1 % :	$-\frac{13}{50}$
„ kal.	1 % :	$-\frac{40}{50}$
„ ammon.	1 % :	$-\frac{11}{50}$
„ plumbic.	1 % :	$-\frac{60}{50}$
„ calcic.	1 % :	$-\frac{10}{50}$
Nitras kalic.	1 % :	$-\frac{60}{50}$

Acids.

Ac. lactic	$\frac{1}{8}$ % :	$+\frac{20}{50}$
„ acet.	$\frac{1}{8}$ % :	$+\frac{35}{50}$
„ „	1 % :	$-\frac{10}{50}$
„ hydrochl.	$\frac{1}{8}$ % :	$-\frac{30}{50}$
„ citric.	1 % :	$-\frac{20}{50}$
„ amygdal.	$\frac{1}{2}$ % :	$-\frac{40}{85}$
„ tartaric.	$\frac{1}{2}$ % :	$-\frac{70}{100}$

Bases.

Sol. NaOH	1 % =	$-\frac{10}{50}$
„ KOH	1 % :	$-\frac{40}{50}$
„ NH_4OH	$\frac{1}{4}$ % :	$+\frac{15}{30}$

Here again we notice the lyotrope series, as with spray-electricity, at all events for the anions series:

Sulfas	Kalic.	$\frac{1}{2}$ N. :	—30
Phosph.	„	$\frac{1}{2}$ N. :	—30
Citras	„	$\frac{1}{2}$ N. :	—25
Chloret.	„	$\frac{1}{2}$ N. :	—22
Nitras	„	$\frac{1}{2}$ N. :	—15
Acetas	„	$\frac{1}{2}$ N. :	—13
Rhodan.	„	$\frac{1}{2}$ N. :	—12
Iodet.	„	$\frac{1}{2}$ N. :	—10

In this list only acetate occupies too low a place, as compared with the lyotrope series in the case of spray-electricity. Of the other substances that proved to be active in spraying, we investigated some glucosides and saponins, antipyretica and alkaloids, which, if not examined as a salt, invariably heightened the charge.

Glucosides and Saponins.

Aesculin	$\frac{1}{800}$:	$+\frac{75}{50}$
Saponin	$\frac{1}{800}$:	$+\frac{70}{50}$
Digitalin	sol. :	$+\frac{65}{50}$

Antipyretica.

Pyramidon	$+\frac{90}{50}$
Antipyrin	$+\frac{100}{50}$

Alkaloids.

Caffein (dil.)	$+\frac{100}{50}$
Theophylin (dil.)	$+\frac{60}{50}$
Hydrochl. morphini	$+\frac{50}{20}$

Lastly, all colloidal substances examined, appeared to increase waterfall-electricity, even an albumin-solution, which always contains salts, provided the solution be dialysed.

Colloidal substances.

Gelatin $^1/_6$ % sol.:	+ $^{70}/_{50}$	Extr. quassica $^1/_{800}$:	+ $^{100}/_{75}$
Tragacanth (very dil.):	+ $^{65}/_{50}$	Glucocholz. natr. $^1/_{1000}$	+ $^{80}/_{50}$
Traganth (dil. sol.):	+ $^{70}/_{50}$	Albumin ov. sicc. $^1/_4$ %	+ $^{20}/_{30}$
Amyl. oryzae $^1/_8$ %:	+ $^{75}/_{50}$	,, ,, dialysed	+ $^{35}/_{30}$
Dextrin $^1/_5$ %:	+ $^{75}/_{50}$		
Gummi arabic $^1/_{600}$:	+ $^{40}/_{35}$		
Fel Tauri $^1/_{1000}$:	+ $^{100}/_{40}$		

It goes without saying that with stronger concentrations the deflection diminishes in consequence of an increased viscosity.

As said, the influence of the temperature on waterfall-electricity is great.

Tapwater of 8° : 40
,, ,, 35° : 100.

Rise of temperature, therefore, increases the positive charge of water exceedingly.

The influence of addition of a saltsolution is also a fact that cannot be denied. Alcohol, which, without salt, increases the positive charge in water considerably, partly loses this faculty at first in the presence of common salt, and even loses it altogether with a concentrated salt-solution. Then the negative charge of the salt predominates. Salt added to the negatively electrifying camphor heightens this negative charge. Camphor and salt co-operate. It should seem then that in the case of waterfall-electricity a simple summation takes place of the effects of water, salt and the volatile addition [1]).

In endeavouring to account for these phenomena we might look upon spray-electricity as well as upon waterfall-electricity as a form of frictional electricity. In either case the friction, between the liquid and the air in the outflow of the spray, between liquid and zinc plate in the waterfall, would set free electrons that are scattered about in the surrounding air. But then the liquid were invariably to be charged positively, which turns out differently in a majority of cases, as shown by the experiments. We presume, therefore, that a more intricate process is at work, in which larger corpuscles arise as carriers of the electric charge. Such formations might perhaps arise from the so-called ions, an equal number [2]) of positively- and negatively-charged ions, round

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[1]) In the case of spray- electricity the process is a much more complicate one. (See E. L. Backman, Researches Physiol. Lab. Utrecht (5) XIX p. 210.

[2]) H. Zwaardemaker. "Le phénomène de la charge des brouillards de substances odorantes. Arch. Neerl. Physiol. de l'homme et des animaux" Tome I 1917 p. 347.

which the water vapour may condense into droplets and with which salt-droplets may combine afterwards.

LENARD [1]) believes that *in* the superficial layers of every di-electric liquid there is not only an electric double-layer, generated by the molecular forces of the liquid itself, the negative layer being situated on the outside, but also that these layers differ as to material.

These differences, which vary with the substances dissolved in the liquid (electrolytes, volatile substances, complex molecules) affect the thickness and the strength of the electric double-layer.

It appears then that LENARD reduces the problem of the origin of waterfall-electricity and of spray-electricity to his hypothesis regarding the specific condition of the surface of every di-electric liquid.

Strictly the origin of the electrification would then be, not an emission of electrons, but a discharge of extremely fine droplets, the so-called "carriers", which, varying with the surface condition of the liquid, are either very small and charged negatively, because they take their origin entirely from the outer negatively charged layer of the liquid, or they are somewhat larger and may be positively charged, since the majority of them arise from the interior positive layer of the liquid.

For further particulars we refer to LENARD's article itself.

Suffice it to state that most of our results with waterfall-electricity are sufficiently explained by this theory.

Not however the intensifying influence of rise of temperature (LENARD's private opinion, founded on theoretical considerations, was that a lessening influence was to be looked for).

Neither does this theory explain why camphor and bornylacetate diminish waterfall-electricity; no more is the question of the intensifying action of the sweet substances and the colloidal substances settled by it.

The results obtained before in the Utrecht Physiological Laboratory in experiments on spray-electricity [2]) are much less easy to explain with the aid of this theory.

First of all pure tapwater (Utrecht Water Company) and *fresh*-distilled water (old-distilled water *is* active) give no or an inappreciable charge in spraying.

Secondly the intensification of the charge in consequence of addition

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[1]) P. LENARD. Ueber Wasserfallelektricität. Ann. der Physik. Bd. 47—1915.

[2]) H. ZWAARDEMAKER. Het in overmaat geladen zijn van reukstofhoudende nevels. Verslagen K. A. v. Wetensch. Deel XIX N°. 1.

H. ZWAARDEMAKER. Specifieke reukkracht en odoroscopisch ladingsverschijnsel in homologe reeksen. Id. Deel XIX N°. 2.

of an active substance, is generally much greater with spraying than with the waterfall.

Thirdly, according to LENARD's theory, salt-solutions yield a distinct negative waterfall electricity; on the other hand they did not give a charge with spraying [1]).

Finally LENARD's theory proves the bigger positive carriers of a common salt solution to contain sodium; the smaller negative carriers on the contrary consist of pure water molecules. This he demonstrated by bringing the carriers between the plates of a condenser, in which process the bigger positive Na-containing carriers went over to the negatively charged condenser-plate where the presence of sodium could be proved. The smaller negatively charged carriers went over to the positive plate, where no sodium could be found.

A similar experiment with *sprayed* salt-solution [2]) shows that the big, positive carriers as well as the small negative ones contain sodium.

We conclude, therefore, that though there is a correlation between waterfall-electricity and spray-electricity, they are obviously not quite identical.

H. ZWAARDEMAKER. Reukstofmengsels en hun laadvermogen door nevelelectriciteit Id. Deel XXV 30 Sept. 1916.

H. ZWAARDEMAKER. Le sens de l'adsorption des Subst. Volatiles Acta Oto-laryagologica.

H. ZWAARDEMAKER en H. ZEEHUISEN. Over het teeken v. h. ladingverschijnsel en de bij dit verschijnsel waargenomen invloed op de lyotrope reeksen. Verslagen K. A. v. Wetensch. deel XXVII 1918.

E. L. BACKMAN. De olfactologie der methylbenzolreeks. Id. Deel XXV 27 Jan. '17.

E. L. BACKMAN. Ueber die Verstäubungselektricität der Riechstoffe. Arch. f. d. ges. Phys. Bd. 168 S. 351.

C. HUYER. De olfactologie v. aniline en homologen. Diss. "Onderz." (5) Deel XVIII, p. 1, 89.

[1]) Afterwards we succeeded in demonstrating a slight negative charge also with spraying a 1 % common salt solution by lessening the capacity of the receiving disc. This charge, however, is not nearly so great as the one evoked in the waterfall and is far inferior to the spray-electricity generated by additions of volatile substances and of substances that activate the surface.

[2]) A. STEFANINI and G. GRADENIGO. Inhalazione di Nebbië Salina Secche. Lucca 1914, p. 22.

Physiology. — *"The Action of Atropin on the Intestine depending on its amount of Cholin"*. By Dr. J. W. Le Heux. (Communicated by Prof. R. Magnus).

(Communicated in the meeting of September 27, 1919).

The action of atropin on the intact, isolated small intestine of mammals has been the subject of a considerable amount of experiments, but no explanation, could be found for the various and widely different results achieved by the several authors. [1])

While Magnus [2]) established that the isolated intestine of the cat, in Ringer's solution, is paralysed in large doses of atropin (0,3 %), but mostly reacts on moderate quanta of atropin (0,025 –0,075 %) with symptoms of stimulation, the pendulum movements, executed by the intestine, getting more regular, especially when the movements of the intestine were previously insignificant, also an inhibiting effect of very small atropin-doses (0,005—0,05 %) was demonstrated by Unger [3]). Other researchers also sometimes found this (paralysing) effect, sometimes they did not. In the isolated small intestine of rabbits and dogs Kress [4]) noted stimulation consequent on moderate amounts of atropin, paralysis through large quanta, whereas others again obtained widely varying results. In most cases small quantities of atropin caused inhibition in the small intestine of rabbits, whereas moderate atropin-doses alternately stimulated and paralysed the gut without any regularity.

According to P. Trendelenburg [5]) the intact intestine of the rabbit reacts regularly on small quanta of atropin with paralysis, but on moderate quantities now with symptoms of stimulation, now with paralysis.

According to the same writer [5]) an exception to irregular behaviour

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[1]) An extensive survey of the literature is given by G. Liljestrand, Pflüger's Archiv, Bd. 175, p. 111, 1919.

[2]) R. Magnus, Versuche am überlebenden Dünndarm von Säugetieren. I. Mitt. Pflüger's Archiv. Bd. 108, pag. 1, 1905.

[3]) M. Unger, Beiträge zur Kenntnis der Wirkungsweise des Atropins und Physostigmins auf den Dünndarm von Katzen. Pflüger's Archiv. Bd. 119, pag. 373, 1907.

[4]) K. Kress, Wirkungsweise einiger Gifte auf den isolierten Dünndarm von Kaninchen und Hunden. Pflüger's Archiv. Bd. 109, pag. 608, 1905.

[5]) P. Trendelenburg, Physiol. u. Pharmacol. Untersuchungen über Dünndarmperistatiek. Schmiedeberg's Archiv. Bd. 81, pag. 55, 1907.

of the gut of various species of animals towards atropin is afforded by the small intestine of the guinea-pig, which is regularly inhibited by atropin, a question to which we will revert in this paper.

It is evident from this short survey that the behaviour of the intestine of different mammals towards small and moderate quanta of atropin is varying and inconstant. No doubt the researchers who obtained these various results, have been working under incongruous circumstances. As yet no one has succeeded in accounting for their conflicting results.

From experiments by v. LIDTH DE JEUDE [1]) it appeared distinctly that the explanation is not to be looked for in the different composition of the salt-solution, in which the isolated intestine was examined.

LILJESTRAND [2]) showed that the various results could neither be ascribed to the different composition of the atropin-preparations; he is rather inclined to believe that the explanation can be found in the gut itself. The object of the present paper is to verify this conception.

WEILAND [3]) has demonstrated that from the stomach, the small intestine and the large intestine a coctastable substance can be abstracted through extraction with water, which has the property of urging on the movement of the gut, and that this stimulating action can be arrested antagonistically by small quantities of atropin. It afterwards turned out [4]) that this substance consists for the greater part of cholin and that it can be obtained from the small intestine of the rabbit to such a quantity that it must act a prominent part in evolving the automatic intestinal movements.

Now cholin belongs pharmacologically to the group of pilocarpin; the stimulating effect exercised by cholin, just as by pilocarpin, on the gut is antagonised by slight quantities of atropin.

VAN LIDTH DE JEUDE [5]) showed by his experiments that the quanta of atropin required to check the pilocarpin-action upon the gut, are very slight; already a concentration of atropin of 1 to 10—50 million will do. Now I found that the atropin-concentrations, necessary

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[1]) A. P. v. LIDTH DE JEUDE, Quantitatieve onderzoekingen over het antagonisme van sulfas atropini enz. Thesis. Utrecht 1916.

[2]) G. LILJESTRAND, l.c.

[3]) WEILAND, Zur Kenntnis der Entstehung der Darmbewegung. Pflüger's Archiv. Bd. 147, pag. 171, 1912.

[4]) J. W. LE HEUX, Choline als Hormon der Darmbewegung. Pflüger's Archiv. Bd. 173, pag. 8, 1918.

[5]) A. P. v. LIDTH DE JEUDE, l.c

to induce a temporary inhibition of the intestinal movements, fall within the same limits.

When combining these facts, the question arises whether perhaps the inhibition of small atropin-doses is to be considered as an antagonism for the cholin present in the intestinal wall.

If this is the case, the inhibition of small quantities of atropin, will not appear when the cholin has been previously removed from the intestinal wall, but it will come forth again after the addition of cholin and subsequent administration of atropin.

Aside from this inhibition of small quantities of atropin, its actual influence is, according to MAGNUS's [1]) experiments, one that stimulates AUERBACHS's plexus. It is only large quantities that paralyse the nerve centra and muscles of the intestinal wall.

The action of these latter quantities we leave out of consideration.

Experimental evidence in support of the above hypothesis may be obtained in the following way:

1st A gut, inhibited originally by small quantities of atropin, is to be brought into a condition in which a small quantity of atropin is without effect, through repeated washings, so that cholin is removed from the intestinal wall.

2nd The atropin-effect is to reappear in this gut after giving cholin.

3rd A gut, which is originally inhibited by moderate quantities of atropin, is to be brought, through repeated washing, into a condition, in which the same quantity of atropin has only a stimu- appeared lating effect.

The experiments made to prove this, were performed with the isolated small intestine of rabbits and guinea-pigs, which, as had before, are provided with rich quantities of cholin and — as may be expected, readily give them off to the environing fluid.

Experiments with the small intestine of the rabbit.

In the experiments with the small intestine of the rabbit a difficulty arose in that after some days the spontaneous movements diminished with the washing out of the cholin, which made the results less clear.

Turning to account LAQUEUR's [2]) experience that loops of intestine

[1]) R. MAGNUS, Versuche am überlebenden Dünndarm von Säugetieren. V. Mitt. Pflüger's Archiv. Bd. 108, pag. 1, 1905.

[2]) E. LAQUEUR, Over den levensduur van geisoleerde zoogdier-organen met automatische functie. Verslagen Kon. Akademie v. Wetenschappen te Amsterdam. 24 April 1914, XXII, p. 1318.

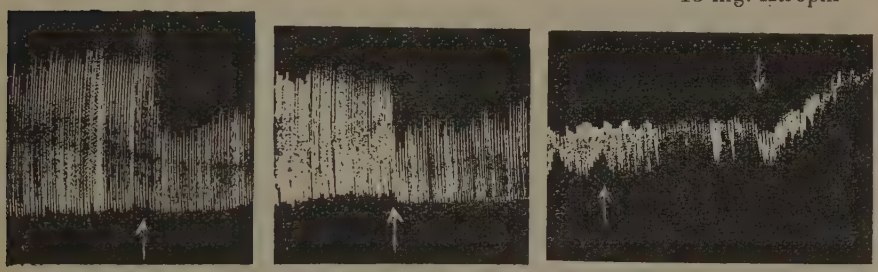
kept in horseserum at a low temperature, retain mobility for days, we now proceeded as follows:

A certain number of loops were severed from the fresh small intestine, which had been cautiously cleaned with Tyrode solution; their movements were registered by MAGNUS's method. The vessels containing the loops were filled with 75 c.c. Tyrode solution of 38° C.

When small quantities of atropin (0,002—0,01) were added, the pendulum movements of the loops got invariably smaller.

This is illustrated in Fig. 1.

Fig. 1. Fig. 2. Fig. 3.
15 mg. Atropin



0.01 Atropin 15 mg. Atropin 0.01 Atropin

Fig. 1.

Pendulum movements of the rabbit's small intestine suspended directly after killing the animal. By administering 0,01 mgr. of atropin the magnitude of the movement is reduced by half.

Fig. 2.

The loop whose movements are registered here is the same as in fig. 1. The previous action of the atropin (0.01 mgr) is entirely eliminated through washing three times with Tyrode solution. 15 mgr. of atropin now yields a strong inhibition on the movements while the tonus is only slightly lessened.

Fig. 3.

A loop of the same gut, standing for 3×24 hrs. in the refrigerator in horseserum that was repeatedly refreshed. Subsequently the loop is washed out eight times with warm Tyrode solution and no longer reacts on 0.01 mgr. of atropin. After 15 mgr. of atropin a marked stimulation appears together with a considerable increase of tonus.

After this the loops of intestine were washed out with fresh Tyrode solution three times every 10 minutes, by which, as we know from our own experience, the effect brought about by the preceding small atropin-dosis was again completely eliminated.

When thereupon a moderate dosis (15 mgr.) of atropin was added, the pendulum movements became considerably smaller in far and

away most cases, while at the same time the tonus was more or less lowered.

This is exemplified in Fig. 2. In a few cases the initial inhibition, caused by this moderate quantity of atropin, was followed by a rather strong stimulation.

This occurred, when the loops had already been cleaned *several* times, so that the gut had reached a stage, in which the inhibition of moderate atropin doses passes into a stimulating action. The next day some loops were again severed from the gut which had been kept standing during the night in horse-serum at a low temperature; after having been carefully washed free from the adherent serum they were suspended as before. Every ten minutes these loops were cleaned with fresh Tyrode solution. One of them (we always took the same) was experimented on to ascertain whether 0.01 mgr. of atropin still evolved inhibition on the movements. If it did not, the other loops of intestine were cleaned again some times and examined with regard to their behaviour towards atropin.

In the great majority of cases it appeared again that a small amount of atropin (0,01 mgr.) did not cause the slightest change in movements or tonus, but that after administering 15 mgr. of atropin a stimulation of the intestinal movements together with a large increase of tonus was noticeable. See Fig. 3. Over and beyond all this the primitive inhibition of the atropin could be elicited again in this stage of the experiment, if a small amount of cholin (1—2 mgr.) had previously been added to the loops.

We did not always succeed in reaching this stage already on the second day, so that it proved necessary to keep the gut in the repeatedly refreshed horse-serum some days longer, in order to arrive at a condition in which small doses of atropin do not affect the gut, which again had been cleaned repeatedly with Tyrode solution.

We also succeeded in obtaining this condition by merely cleaning the gut with Tyrode solution, i.e. without the appliance of horse-serum. It is true, though, that, as mentioned before, the movements will become smaller then, and the results less clear. This proves, however, that the results are not influenced by horse-serum.

In the foregoing we have thus shown for the rabbit's small intestine:

1st. that repeated washing, which, as demonstrated before, deprives the intestinal wall of cholin, evolves a condition in which the initial inhibition of small amounts of atropin, is arrested.

2nd. that by administering cholin this inhibition of atropin may be elicited again.

3rd. that the effect of moderate quantities of atropin, which was variable at first and in my experiments was mostly inhibitory, may be altered in a constant, stimulating effect by repeated washing.

Experiments with the small intestine of the guinea-pig.

We now took the small intestine of the guinea-pig as the object of our investigation.

Recently TRENDELENBURG [1]) has suggested an effective method to register graphically the peristaltic movements of the surviving small intestine of the guinea-pig and to determine to a certain extent in numerical values the action exercised on these movements by various poisons.

TRENDELENBURG records that atropin (acting on the small intestine of the guinea-pig) is *invariably* inhibiting peristalsis.

The great thing in our experimentation was to ascertain whether here also, as with the rabbit's small intestine, its behaviour towards atropin is governed by its condition. We used TRENDELENBURG's method and proceeded as follows:

The guinea-pig was killed by a blow on the neck, the small intestine was cautiously severed from the mesentery, and cleaned several times, with a warm fluid [2]) after LOCKE. Subsequently the intestine was cut into 5 parts, one of which was suspended immediately, the other pieces were put in separate dishes with LOCKE's solution, which was refreshed every now and then.

The loop of intestine which was suspended in a vessel of 150 c.c. capacity, was first left to itself with an interior pressure of 0 mm. H_2O.; then the pressure was gradually heightened and we determined at what pressure peristalsis first appeared (critical pressure). Then the interior pressure was lowered to 0 and after 3 minutes the critical pressure was again determined.

This determination was repeated after 0,1, 1, and 5 mgr. of atropin had been added respectively. In accordance with TRENDELENBURG's report, arrest of peristalsis took place, so that no peristalsis occurred any more even when the pressure was made considerably higher.

After being carefully cleaned, a second loop was suspended, which was kept standing for some hours in LOCKE's solution; the same determinations were made prior to and posterior to the administration of atropin. In most cases it appeared already now that the

1) TRENDELENBURG, l.c.

2) It is essential that the fluid should be prepared from pure salts and with pure water distilled from glass apparatus.

small atropin-dosis yielded a much weaker inhibition on the peristalis than with the first loop.

Subsequently another loop was examined, which had been kept standing in LOCKES's solution some hours longer again, and had been washed a few times more etc.

Table I comprises the results of the complete experiment.

From it we see that the gut, which is arrested by quantities of 0,1, 1, and 5 mgr. of atropin, is brought after a 2½ hours' washing with LOCKE's solution into a condition in which 0,1 and 1 mgr of atropin produces a *much weaker* inhibition.

TABLE I.

Number of the loop of intestine.	Time of washing.	Critical interior pressure in mm H_2O.			
		normal.	after 0.01 mgr. of atropin.	after 1 mgr. of atropin.	after 5 mgr. of atropin.
1	0	7	Inhibition	Inhibition	Inhibition
2	2,5	20	25	30	Inhibition
3	6	20	16	20	Quick 20 pendulum movements.
4	8	12—16	13	—	—
5	11	10	7	—	—

After the process of washing had been prolonged for 6 hours the inhibition of these atropin quantities had entirely stopped, but now peristalsis appears after 0,1 mgr. of atropin already at a lower interior pressure as in the normal period. Also with loops of intestine that have been washed 8 and 11 hours atropin causes distinct stimulation, so that peristalsis comes forth already at a lower interior pressure.

With loop N°. 3 a very considerable increase of the pendulum movements was also perceptible after 5 mgr. of atropin.

It has thus been proved also for the small intestine of the guinea-pig that the inhibition of atropin, already established by TRENDELENBURG, can be arrested by washing and that the atropin-action proper, stimulation of AUERBACH's plexus, can be elicited by small quantities of atropin.

Now it is obvious also why TRENDELENBURG noted only the inhibition of atropin. It was because he did not allow the loops to stand in a solution, but always experimented with fresh ones taken from the guinea-pig under urethannarcosis.

The results obtained by our experiments put us in a position to view the variable action of atropin on the gut in a new light and to interpret the conflicting results of the various researchers.

In the living animal cholin is present in the intestinal wall in such quantities that they stimulate AUERBACH's plexus. On removal, to a certain extent, of the cholin from the surviving gut by a prolonged washing, the real action of atropin manifests itself distinctly. According to the earlier experiments of MAGNUS [1]) it consists in a stimulation of AUERBACH's plexus by moderate quanta, whereas only very large doses paralyse the centra, the nerve, and the muscle. Originally the intestinal movements are not affected by small quantities of atropin.

It is, therefore, upon the presence of more or less cholin in the intestinal wall that the atropin-action depends.

Cholin has a stimulating effect upon AUERBACH's plexus, which is antagonised by atropin. So long as an adequate quantity of cholin is present in the intestinal wall, a small dosis of atropin will inhibit the cholin action and consequently inhibit the intestinal movements.

The result of the action of moderate quantities of atropin will depend on the circumstance whether the immediate stimulating action on the plexus, or the antagonism for cholin preponderates.

In case the gut contains little cholin the stimulating action comes to the front, in case it contains much cholin the antagonism (inhibition) predominates. With a moderate cholin-content a stimulation will succeed an initial inhibition.

Likewise we are now enabled to account for the results of earlier researches.

The fact that with the cat's small intestine the stimulating effect of moderate quanta of atropin occurs more often than with the rabbit's or the guinea-pig's, tallies with our experience that the former contains less cholin than the latter two.

On the other hand it stands to reason that with the guinea-pig gut, which was always found to be rich in cholin, the inhibition of atropin appears regularly.

It is obvious now why the isolated rabbit's gut, according to the previous treatment, is now inhibited by atropin, now again is stimulated, while on the other hand the intact gut, which could not be liberated from cholin by washing, is according to TRENDELENBURG [2]) inhibited regularly.

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1) R. MAGNUS, l.c.

2) P. TRENDELENBURG, l.c.

Summary.

We have demonstrated in an earlier paper that the isolated small intestine yields to salt-solutions quantities of cholin, which are capable of stimulating AUERBACH's plexus.

This loss of cholin results in a changed behaviour of the gut towards atropin.

The rabbit's small intestine that is inhibited previous to washing by small doses of atropin, no longer reacts on them; on the other hand it is now stimulated by moderate doses.

The normal guinea-pig gut is invariably inhibited by atropin. This effect also here disappears after washing and is substituted by a stimulation through moderate quanta of atropin.

This is to be interpreted as follows: the real action of moderate quanta of atropin on the gut is stimulation of AUERBACH's plexus; if the gut contains much cholin, so that the plexus is readily stimulated, this stimulation is arrested through the antagonism of a small amount of atropin, occasionally weakened, and the result is inhibition. Moderate quanta of atropin are also inhibitory when this antagonism is strong enough, but in the presence of small quanta of cholin in the gut the latter will be stimulated.

It is clear, therefore, that here we have to do with a case in which the presence of a well known chemical substance (cholin) in the tissue determines the manner in which this substance reacts on a poison (antropin).

The Pharmalogical Institute of the Utrecht University.

Sept. 1919.

Zoology. — *"The interrelations of the species belonging to the genus Saturnia, judged by the colour-pattern of their wings."* By Prof. J. F. VAN BEMMELEN.

(Communicated in the meeting of May 3, 1919).

In his handbook of Palaearctic Macrolepidoptera STANDFUSS says on p. 106, at the end of his passage on the relative age of the three species Saturnia spini, pavonia and pyri: "If we exclusively paid attention to the imagines, much might be said in favour of the opinion, that spini is older than pavonia. For both sexes of spini, and likewise those of the new species from Kasikoparan (S. Cephalariae Ch.) discovered as late as 1882, possess a remarkably uniform type. From this type the male of S. pavonia sharply deviates, this species thereby appearing as recently changed in its imaginal dress. But as to the question if we should place the origin of Sat. pyri before the evolution of these minor forms or after it, the imago of this latter species does not seem fit to allow of a really certain conclusion.

In regard to the caterpillar- and pupal-stage, however, things are different. For these it can be clearly proved that S. spini, pavonia and pyri form in many instances three different degrees of protective resemblance against certain hostile factors of the outer world. In this scale everywhere spini occupies the lowest, pyri the highest degree. Keeping in view the excessively near relation and the great similarity of biological conditions between the three species, we are obliged to assume that spini came into existence before pavonia, and pavonia before pyri, or, using the scientific expression for these relations: that phylogenetically spini is the oldest, pavonia a younger, pyri the youngest form. For it evidently would be absurd to assume that in a series of so intimately related forms, the more perfect living being should have originated at an earlier date than the less perfect one".

These considerations of STANDFUSS induced me to compare the wing-markings of Saturnine-species, as well among themselves as with those of related genera, to see if this line of investigation did or did not lead to corresponding results as the inspection of the caterpillars.

In accordance with the general rules for the character of the wing-markings, which I thought myself justified in proclaiming, and which I tested as to their applicability to such families as Hepialids, Cossids, Arctiïds and Sphingids, I came to the conclusion that not pyri but pavonia should be considered as the oldest form. For in pavonia the festooned submarginal transverse lines and bands deviate less from the outer wing-margin and also show a smaller difference between their anterior and posterior extremity, the submarginal band therefore having the simplest and least irregular type. In the same way the difference between fore- and hindwing, as well on their upper- as on their underside — and therefore also between the superior and inferior surface of each of the wings for itself —, is smaller in pavonia than in the two other species.

In comparison to other Bombycids and to the remaining families of Heterocera, the colour-pattern of pavonia shows a greater similarity to the general primordial pattern of seven dark transversal bars, which I deduced from the comparison of all these forms, than the two other Saturnids. It might seem that this assertion is contradicted by the fact, that in pavonia the male at first sight looks entirely different from the female by its colour as well as by its inferior size, a difference which has apparently made a deep impression on STANDFUSS. But on nearer inspection and consideration the difference is by no means so important as it looks, and need not be regarded as of high importance. For the difference in hues is evidently connected with the frequently occurring feature of discoloration (i. e. partial self-colour) by which on the superior surface the hindwing has partially turned into yellow, while at the underside the same hue has spread over the proximal part of the forewing. On both wings this discoloration is accompanied by a slight and incomplete fading of the pattern.

Moreover it may be doubted, on very sound arguments, if the yellow hue — apart from its spreading over the dominion of spots and stripes, which are rendered more or less invisible by it — should be considered as a secondary modification of an older and more original hue, which latter therefore should have persisted on the upperside of the forewing and on the underside of the hind one. For this yellow-brown hue is characteristic for quite a number of Bombycine moths belonging to different genera, and in so far impresses us as a very original colour. It might therefore be assumed, that its occurrence in the male of pavonia should be considered as a reversion to an older condition, instead of being the appearance of a new hue.

But we may safely leave these questions unheeded, for the hues in which a pattern is executed, need not be taken into consideration, neither in this case nor in others, when judging the character of the pattern itself.

Now, if we carefully compare the markings of the male with those of the female and pay due attention to the half-vanished stripes and spots under the overspreading yellow, it becomes clear, that down to the minutest details, male and female agree in pattern. No more is there any necessity to consider the sexual dimorphism as an important progressive feature, by which p a v o n i a should distinguish itself from s p i n i and p y r i, and be characterised as a recently and strongly modified form, in contrast to the other two, which would have remained more conservative. For secondary and tertiary sexual differences occur in all kinds of Lepidoptera, as well as in other insects. In the Bombycine moths this feature shows itself in a remarkably high number of forms. Now have we really to consider each of these cases as a separate and independent deviation from a common original condition, in which male and female were alike in shape, size, hues and pattern? Or did the phenomenon of sexual dimorphism already show itself amongst primitive Bombycidae, in the days when the difference between them was less considerable than at present, and they still counted fewer specific and varietal forms; dimorphic species therefore then existing side by side with monomorphic as well as now. In this case the sexual dimorphism of p a v o n i a might repose on the manifestation or the permanency of an old hereditary disposition.

But even apart from this question, which can hardly be solved with certainty, it remains doubtful, if the species p y r i and s p i n i should really be considered older than p a v o n i a, on account of the similarity of their sexes. For I by no means consider it as proved, that in p a v o n i a the male, which deviates from the general hue of the genus, may be considered as the modified form, while the female, which seems to show so much more similarity to s p i n i and p y r i, may be regarded as the unchanged form.

When the general rules for the colour-pattern are blindly applied the solution of this question might seem easy enough. In the female of p a v o n i a as well as in both sexes of s p i n i and p y r i the fore- and hindwings on their upper- as well as on their underside, show the same clear whitish hue, here and there overspread with a black sprinkling, and subdivided into fragments by nervural and transversal straight or undulating lines. But this

general clear tint, reigning over the whole extension of the wings, as well as the similarity in design between fore- and hindwings, upper- and underside, impresses me as a secondarily acquired uniformity, rather than as a really original feature.

In my opinion we meet here with a similar case as in the wholly selfcoloured butterflies e. g. Gonepteryx rhamni or Aporia crataegi, in which without the least doubt the uniform hue is the consequence of the simplification of the shades, hand in hand with the total or partial regression of the markings. Still more striking is the similarity with the Parnassine butterflies (which, it need hardly be said, is of course wholly superficial and occasional). For in these as well as in Saturninae a set of highly differentiated eye-spots on the forewings, but still more pronounced on the hindwings, form the most conspicuous part of the pattern, though it may undoubtedly be taken for granted that the spotted design of the Parnassines has developed from a far more complete array of simpler and more uniform markings, such as are seen in Thais polyxena and its consorts.

As soon as we consider the similarity of fore- and hindwing, upper- and underside of the female of pavonia and of both sexes of pyri and spini as secondarily acquired characteristics, there need no more be any objection against the supposition that the male of the first-named species is more original than the female.

From this point of view we may further remark, that the existing contrast between the yellow upperside of the hindwing and the greyish one of the forewing, on which the markings are less hidden under the ground-colour, corresponds in a higher degree to the general type of butterfly-design than does the exact similarity of fore- and hindwing; the overwhelming majority of Lepidoptera showing a similar difference between fore- and hindwing.

In the fact that the discoloration (in this case yellowing) occurs on part of the *upper*side of the *hind*wing and of the *under*side of the *fore*wing, the male of pavonia is in harmony with both sexes of Smerinthus ocellata and of many other Lepidoptera. Up to a certain degree the frequent occurrence and similar extension of this discoloration can be attributed to the influence of the resting attitude of these moths, the discoloured areas being exactly those which during this attitude remain covered by similar parts of the adjoining wing. In his paper: "Sur la position de repos des Lépidoptères", J. T. Oudemans has directed our attention to this circumstance and expressed his opinion that during the development and modification of the colour-pattern in the course of time, the

Fig. 1. SATURNIA BOISDUVALII ♂.

Fig. 2. OPHIDERES DIVIDENS ♂.

Fig. 3. NYCTIPAO CREPUSCULARIS ♀.

Fig. 4. EMMONDIA PUNCTATA ♀.

E. THEYSSEN, phot.

hidden and the exposed parts had independently proceeded each along its own course (p. 81—83), on which the one as well as the other could get on at a higher speed and so reach a stage more remote from the original common condition.

From a general point of view I feel inclined to join this opinion, but in the case at hand it brings us little light. The difference between the covered and the exposed wing-areas in the attitude of rest, already slight in the male of pavonia, is quite insignificant in the female, as well as in both sexes of pyri and spini. The wing-markings on these areas seem not to be influenced to any notable degree by the habit of passing the forewings over the greater part of the hindwings during day-time. That traces of such an influence are still visible in the male of Sat. pavonia, might be taken as an indication that the original influence of the said habit is now gradually losing its force.

The highly conspicuous eye-spots in my opinion must have evolved from simpler discoidal marks, hand in hand with the above mentioned change in the influence of the resting attitude. This is already proved by their very different degree of differentiation in the several species of the genus Saturnia and of kindred genera. Eye-spots moreover always are special differentiations, secondarily developed on the base of a more primitive and simple colour-pattern, whose elements have occasioned them by modifications in the original shape, colour, size and direction.

In my previous paper, on the wing-markings of Sphingides, I tried to prove this assertion for the case of Smerinthus ocellata, in the same way as I formerly did for the Hepialid moth Zelotypia stacyi. It must however be possible to prove it as well in other cases, e.g. for Vanessa io and many other Nymphalids, and likewise for Satyrids and Lycaenids. Cynthia for instance shows in what way eye-spots and simple spots alternate in the row of submarginal markings, and also often how an eye-spot on the superior surface is represented by a common one on the underside. The difference between the two seasonal forms in the first place depends on the contrast in the differentiation of the eye-spots: in the dry-monsoon-generation they are scantily developed, in the wet-dito highly so.

Now, do the hybridisations of Standfuss throw any light on these questions? As far as I can see not much; in general the hybrids are intermediate forms, but as to the width of the submarginal dark seam, they agree more with pavonia than with spini, and assuredly far more than with pyri.

But in my opinion in order to judge about the interrelations of the colour-patterns of Saturnia, it is absolutely necessary to compare with each other as many different forms as possible, just as in the case of other groups of Lepidoptera. It certainly cannot stand criticism to draw consequences from the exclusive consideration of three intimately connected species as to the relative age of their colour-patterns.

When pursuing this broader way, the well-foundedness of the above-mentioned assertion, viz. that the colour-pattern of Saturnids is a special case of that of Bombycids, and the latter again of that of Heterocera in general, is clearly proved. To begin with: next to the three above mentioned species stands Saturnia (Caligula Jordan) boisduvalii, on whose upper side the submarginal dark seam broadens from before backwards in still higher degree than in pyri, which reduces the somewhat median clearer area under the eye-spot, (broadest in pavonia) to almost nothing by forcing it back in proximal direction.

The two dark borders by which this area is limited (and which I suppose to be V and VI) are in one place locally connected by a black transversal link. This part of the pattern of boisduvalii therefore shows the greatest similarity with the corresponding area of the pattern of the male pavonia, but in the latter the connecting link seems to run between IV and V. Generally speaking, it is not easy accurately to make out the exact consecutive number of the bands for each separate form, yet the comparison of the superior surface of the different species leaves the general impression, that the posterior broadening of the dark submarginal area is brought about by the progressive darkening of the colour in a proximal direction, which consecutively incorporates the dominion of a more proximally situated transversal bar. In the female of pavonia this darkening process is restricted to the area of Bar II, in the male it has advanced unto III, in the same way as in both sexes of spini, in pyri it has reached IV, in boisduvalii V.

On the inferior surface the broadening of the dark area proceeds more slowly and more equally over the whole extension of the dark submarginal field. Consequently the underside of spini e.g. resembles both the upper- and the underside of pavonia to a higher degree than its own superior surface. On the hindwing the backward broadening of the dark submarginal border is less pronounced than on the forewing, and this edge is there separated by a light-coloured band over its whole length, from the dark festooned line, that runs along the distal side of the eye-spot.

On the underside however (which is never represented even in large illustrated works) we again meet with the same feature, as mentioned just now for pavonia, spini and pyri, viz. that the course of the transversal bars is far more regular and original than on the upperside, this bringing about a much greater similarity between fore- and hindwing on the first mentioned surface.

I want to draw your attention to a simple dark transverse stripe on the middle of both wing-pairs, almost devoid of incurvations and rather faint. On the forewings this stripe runs along the distal border of the eye-spot, on the hindwings along the proximal one. Regardless of this difference, I think we have to deal in both cases with vestiges of bar IV. To this assumption I am especially led by the comparison of the upperside. On that of the forewing the anterior part of this bar is quite apparent up to the eye-spot, which constitutes a marked difference between boisduvalii and the three first-mentioned species. Past the eye-spot however the posterior part of the bar is wanting, but from the postero-interior border of that spot a black stripe runs across to Bar VI, turning sharply at an angle in the middle of its course and then running parallel to Bar VI, perpendicularly to the posterior wing-border and as far as this latter. The lastnamed part belongs to Bar V, as is proved by its comparison with the upperside of the hindwing, on which the Bars IV, V and VI may be perceived in their full extension, though faint and half-hidden under the hairy coating.

On the underside of boisduvalii therefore the dark submarginal border of both wing-pairs is located between Bars II and III; on the upperside however this is only the case on the hindwing. Bar III may even remain separated from the dark seam as an independent isolated stripe, as is shown in the illustrations of Seitz (Vol. II Pl. 31[a]), in contrast with the specimen at my disposition, where the black internal border was not free from the much broader marginal seam.

In some respects Saturnia (Neoris Moore) schenckii corresponds with boisduvalii, e. g. in the presence on its underside of an extremely faint yet complete transversal stripe, which takes its course along the eye-spots. In this species however no difference exists as to the situation of this stripe on fore- and on hindwing, while the stripe is also present on the upper-side, though in an incomplete and unequal way.

Bar V is here the most pronounced and regular, VI and VII are hardly visible.

The eye-spots have been removed outward, and in consequence

the dark submarginal seam has been reduced, while Bar II and III cannot be distinguished from each other. They are represented by a single dark double-line, which in its anterior part is very much festooned. At the outer side of this double-line a white bar runs along, followed by a particularly broad margin of a light-manilla brown hue.

The comparison with the female pavonia makes it probable, that the dark submarginal edge ought to be situated between the white bar and the dark double-line, and that therefore we may assert that it is absent. Yet this is true only to a certain degree: the broadening of the dark submarginal border is, as we remarked before, a consequence of the advancing of the obscuration over the areas between the succeeding transversal bars in a proximal direction, and this identical process is also seen to take place in schenckii, viz. on the fore-wing between double-line II + III and the dark stripe representing IV.

At the underside of schenckii the pattern is simple, and moreover pale and reduced.

A.o. the eye-spots on the hindwings are much smaller, paler and less complete than at the upper side.

Amongst the many genera near-akin, that are arranged around the Saturnids, a great number of additional arguments may be found for the above-mentioned assertion, that their colour-pattern may be derived from the same scheme of seven dark transversal bars, which proved applicable to Arctiïds.

We only need point out forms like Rhodinia fugax, probably showing II, III, and V or VI, or Laepa damartis (Seitz II, Pl. 32^d), which on its forewing wears I, II, III and V or VI, on its hind-wing I, II, III, IV and V (the last two only in part). Even in such a complicated and special pattern as that of Brahmaeïds it is comparatively easy, with a little attention, to find again the seven primary transversal bars.

However it is not only the ground-plan of the wing-design that may be shown with great probability to be common to numerous and various groups of moths, also the modifications of this plan seem to take place after the same rules among the different families of Lepidoptera. In the present case of Saturnine moths it is the broadening of the submarginal dark border in the direction of the hind margin, that constitutes the principal difference in pattern between the various species mentioned and leads to their arrangement in the sequel: pavonia, spini, pyri, boisduvalii, while the pattern of schenckii seems to be due to a secondary regression of the dark band on its forewings.

This broadening in a caudal direction makes the impression, that an oblique line of separation runs across the forewing from tip to root, dividing it into an antero-interior lighter field and a postero-exterior darker one. Calling this line the *V*-diagonal (because the lines of the two wings in expanded attitude form the letter V) we may state, that this *V*-diagonal-design occurs in numerous Lepidoptera of various families, though always as a secondary modification of the original pattern of transversal bars. In the group of Chaerocampine Sphingides, which I hope to discuss in a future communication, this feature is particularly striking.

We find it however in the same way in the families of Hepialidae, Noctuidae and Geometridae, and likewise in numerous Microlepidoptera. In the overwhelming majority of these cases the *V*-diagonal-pattern is restricted to the upper side of the fore-wing; the underside showing the primitive pattern of transversal bars.

As specially striking examples may be mentioned the Noctuid genera Ophideres, Nyctipao and Emmondia, the Bombycid genus Eupterote, as well as many Geometrids.

The species of Ophideres give rise to the remark that the separation of the forewing-area in an antero-internal and a postero-external part, brought about by the *V*-diagonal-design, in many species e.g. O. tyrannus, salamia and fullonica (comp. Seitz III, Pl. 66) seems to be connected with the resemblance of the entire forewing-design to a withered leaf, this likeness bringing the said species under the category of the leaf-imitating Lepidoptera.

We have to deal here once more with such a feature, as can show itself in many different shapes, and therefore in its real character is evidently independent of the importance it may in some particular cases possess for the establishment of protective resemblance.

It further results from the mutual comparison of different species, that the contrast between the anterior and the posterior part of the wing-surface may be very different in quality as well as in quantity. In some species the forepart is light, the hindpart dark, in others they are nearly alike. In O. materna a transversal design of Cossid-markings (traits effilochés Botke, Rieselung Eimer) spreads over both parts; in the male the *V*-diagonal is present, in the female absent, while on the contrary an *A*-diagonal-design is partly developed as a light-coloured-streak. For besides the *V*-diagonal an *A*-diagonal can be distinguished, running from before and inside to behind and outside in an oblique direction over the wing-surface, often it seems to possess some connection with the division of the

wing into a beam- and a fan-part (Spreiten- und Faltenteil as SPULER has termed them).

In the species Miniodes discolor, near akin to the genus Ophideres, the dark *A*-diagonal separates the orange-coloured forewing-area into two parts: both of them well showing the Cossid markings. A remarkable detail in this case is the uniform pink colouring of the hindwing on its upperside, the under one showing just the reverse: self-coloured forewing, Cossid markings on the hind one.

On the upper-side of Nyctipao crepuscularis (SEITZ III, Pl. 58) markings in three different directions coöperate to form the pattern: 1. transversal markings, especially the white line running from the anterior to the posterior margin through the centre of fore- and hindwing, but also the dark bar near to the wing-root, representing Bar VI or VII, 2. the *V*-diagonal designs of the forewing, 3. a white stripe, forming an obtuse angle with the *V*-diagonal, and running parallel to the *A*-diagonal. Inside the angle, made by the *V*- and the *A*-diagonals, an eye-spot has been differentiated from elements of the transversal markings.

Still sharper these three directions of wing-markings stand out in dark bars against a creamy-white fond on the fore-wings of the arctiïd Areas galactina. So numerous are the cases of *V*- and *A*-diagonal-design, that I will not even venture to summarize them and compare them to each other. As a general result however of my comparative investigations, I feel justified to assert, that everywhere the secondary character of this pattern may be stated with surety, and that in the great majority of cases the underside shows no trace of one of these oblique lines, whereas clear vestiges of the seven original transversal bars nearly always occur, and are also frequently present on the upperside, though generally incomplete. In Rhopalocera the *V*-diagonal-design, when it occurs, is restricted to the underside, and here serves to establish the leaf-imitating character.

Groningen, April 1919.

Botany. — *"The Influence of Light on the Cell-increase in the Roots of Allium Cepa"*. By H. W. BERINSOHN. (Communicated by Prof F. A. F. C. WENT).

(Communicated in the meeting of October 25, 1919).

MRS. DROOGLEEVER FORTUYN—VAN LEYDEN [1]) has found that the cells of young cats increase periodically in such a way, that during the night, the number of karyokineses reaches its maximum, and in the later morning hours and the early afternoon a minimum number is reached. KARSTEN [2]) stated likewise that a periodical karyokinesis takes place in the young buds of Zeamais, which reaches its maximum also during the night. However in the roots of Vicia Faba he did not find any periodicity and so he concludes: "Das Wurzelwachstum entbehrt der Periodizität." During these experiments the plants remained in the dark. Now he tried to influence the periodicity by exposing the young plants to the light of an electric lamp, in which he succeeded. On the other hand he did not trace the influence of light and dark on the cell-increase in the roots. As the root growth is evidently not a periodical one, the influence of light and dark will be most obvious here.

I chose Allium Cepa to experiment upon, because the Alliumcells are easily fixed and stained; because one finds a great number of karyokineses in the roottops and because there are a great many roots, so that it is possible to examine parts of one and the same individual under different circumstances.

At 8 a. m., 11 a. m. and 3 p. m. I took a few roottips away from a germinating onion, which was exposed to full daylight. After that I put the same onion in the dark and left it alone until the next day, then I took a few tips off at 6 a. m., at 12 a. m. and at 6.30 p. m., while the onion remained in the dark. During these two days the temperature differed $^1/_2{}^\circ$ C. (registered with a maximum and a minimum thermometer).

I always took care to take roottips shorter than 25 mm. and of about the same length. The roottips were fixed in sublimate

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[1]) Mrs. DROOGLEEVER FORTUYN—VAN LEYDEN. Proceedings Konink. Akad. Amst. Vol. 19. 1916, p. 38.

[2]) KARSTEN. Zeitschr. f. Botanik 1915, p. 1.

sodium chloride, after passing the different alcohols, they were enclosed in paraffin and then they were cut into series of 10μ and the sections were stained according to HEIDENHAIN's ironhaematoxylin method. I counted the number of nuclei of some of these sections over a length of 1 mm. from the roottop and I fixed the number of mitoses. I took care to count only in the central sections. Table I and II give my results.

TABLE I (in daylight).

Time.	Total number of nuclei.	Total number of mitoses.	Spirema and loose chrom.	Monaster.	Diaster.	Two nuclei.
8 a.m.	4000	—	—	—	—	—
11 a.m.	4345	139	53	53	17	16
3 p.m.	2290	47	26	14	5	2

TABLE II (in the dark).

Time.	Total number of nuclei.	Total number of mitoses.	Spirema and loose chrom.	Monaster.	Diaster.	Two nuclei.
6 a.m.	4702	210	125	66	19	0
12 a.m.	4204	180	147	21	20	4
6.30 p.m.	4043	124	65	29	10	20

In order to compare these figures, I expressed them in percentages in the following tables. KARSTEN takes the average of his countings. In my opinion it is more exact to express these facts in percentages, just as Mrs. DROOGLEVER FORTUYN-VAN LEYDEN does, for it is most improbable that KARSTEN always examined the same number of cells.

TABLE III (in daylight).

Time.	Total number of nuclei.	Total number of mitoses.	Spirema and loose chrom.	Monaster.	Diaster.	Two nuclei.
8 a.m.	4000	0.00 %	0.00 %	0.00 %	0.00 %	0.00 %
11 a.m.	4345	3.19 „	1.01 „	1.01 „	0.38 „	0.37 „
3 p.m	2290	2.05 „	1.13 „	0.61 „	0.21 „	0.08 „

TABLE IV (in the dark).

Time.	Total number of nuclei.	Total number of mitoses.	Spirema and loose chrom.	Monaster.	Diaster.	Two nuclei.
6 a.m.	4702	4.46 %	3.19 %	1.4 %	0.4 %	0.00 %
12 a.m.	4204	4.28 „	3.49 „	0.49 „	0.48 „	0.09 „
6.30 p.m.	4043	3.06 „	1.60 „	0.71 „	0.24 „	0.49 „

As is evident, the number of karyokineses reaches its maximum between 8 a. m. and 11 a. m. (solar time), which agrees with the well-known fact that good cell-divisions in Hyacinth and Allium are found between 10 a. m. and 11 a. m. From 11 a. m. the number of cell-divisions decreases to 2.05 % at 3 p. m.

I found the greatest number of karyokineses with the onion in the dark at 6 a. m., 4.46 %. At 12 a. m. this had slightly decreased to 4.28%, and at 6.30 p.m. the decrease was still greater. At 6 a.m. and at 12 a. m. the maximum number of karyokineses in the dark exceeded the maximum number of cell-divisions found in the daylight, while the maximum number of karyokineses in the light surpassed the minimum number of cell-divisions in the dark only by a slight degree; so that the conclusion seems justified: The number of karyokineses in the rootcells of Allium Cepa increases in the dark, which is stated by KARSTEN [1]) for Spirogyra and other plants.

When we compare in the tables III and IV the number of spirema, loose chromosome stages with the number of monaster stages, then we see in the first table from 11 till 3 o'clock an increase of the number of spirema and loose chromosome stages and a decrease of the monaster stages. This would point to an increase in the number of cell-divisions and nevertheless the total number of mitoses has diminished. We see the same phenomenon on table IV from 6 a. m. to 12 a. m.

By considering spirema, loose chromosomes and monaster as one stage (prophase) the number of nuclei in prophase, in table III at 11 a. m. is 2.02 % and at 3 p. m. 1.74 %, which points to a decrease. The same can be applied to table IV. At 6 a. m. 4.59 % and at 12 a. m. 3.98 % and at 6.30 p. m. 2.31 % is in prophase, so there is a total decrease. In my opinion this fact is a confirmation of the general conception to consider spirema, loose chromosomes and monaster as one stage. [2])

[1]) KARSTEN. Zeitschr. f. Botanik 1918.

[2]) PEKELHARING. Weefselleer, p. 67.

From similar facts, as are contained in table III and IV, it seems also possible to me, to conclude something about the rapidity from prophase to anaphase and from anaphase to telophase. Let us consider table III for that purpose. At 11 a. m. 2.02 % were in prophase and 0.38 % in anaphase. The number of karyokinesis figures in prophase has decreased with 13.8 % at 3 p. m. and the number of cell-divisions in anaphase has decreased with 44.7 %, so the decrease is intenser, that is to say, the transition from anaphase to telophase is quicker than the transition from prophase to anaphase. The same holds good for the onion in the dark during the whole day, but during the day an inversion takes place in such a way that from 6 a. m. to 12 a. m. the transition from prophase to anaphase is quicker than from anaphase to telophase.

Of course these facts are too scanty to draw such far-going conclusions, but the aim of this calculation was only to show that it is possible to learn the relative rapidity. If one wants to undertake such experiments it is necessary in the first place to fix the time of observation much shorter, i.e. one hour or one hour and a half. It is also possible to derive the duration of one cell-division from such tables. When we consider table I we do not find karyokineses at 8 a. m., and at 11 a. m. we find 16 nuclei in telophase. So the cell-division would take about 3—4 hours with Allium Cepa. JOLLY found with Triton the duration of the kariokynesis 2½ hours in the erythrocytes at a temperature of 20° C.

From the table of MRS. DROOGLEEVER FORTUYN—VAN LEYDEN I think I may conclude the duration of a cell-division being 12 hours with a cat, because at 2 p. m. ± 0.23 % nuclei were in prophase and no telophases were stated. Only at 2 a. m. 0.20 % nuclei were seen in telophase for the first time.

When we summarize the results, we see that the roottips of the onion show more cell-divisions in the dark than in the light. Evidently light has a retaining influence. Besides it is probable that the transition process from prophase to anaphase is a slower one than the transition process from prophase to telophase.

By lack of time I could not control these facts any further. To attain this, it would be necessary to make an investigation into the daily oscillations in the number of karyokineses with the onion, if possible the time of observation ought to be as long as possible (3 to 4 days). At the same time the above-mentioned experiment ought to be repeated. One onion suffices for these two experiments. The bulb is cut into two halves and one half is used for the first series of experiments and the other half voor the second experiment.

The two series of experiments are made with parts of one and the same individual. Neither Mrs. Droogleever Fortuyn—van Leyden, nor Mr. Karsten have done this, so the results lose reliability.

Notwithstanding the incompleteness of my investigation, I thought the facts I found, of sufficient importance to be examined further, and for this reason I published this communication. At the same time I make use of the opportunity of thanking Mr. M. W. Woerdeman as well for the incitement to this research, as for the kind assistance lent to me.

Amsterdam. *Laboratory of Histology.*

Physics. — *"The Propagation of Light in Moving Transparent Solid Substances. I. Apparatus for the Observation of the* FIZEAU-*Effect in Solid Substances."* By Prof. P. ZEEMAN.

(Communicated in the meeting of May 3, 1919).

1. As a result of an experiment by ARAGO with a *glass* prism FRESNEL drew up his bold hypothesis on the convection coefficient in 1818. When in 1851 FIZEAU wanted to put FRESNEL's hypothesis to the test, he experimented, however, with *water*, and examined whether or not the velocity of light in standing water differs from that in moving water.

There are many reasons to be adduced for carrying out an experiment, so exceedingly difficult as that of FIZEAU, in the first place with water; it is, however, also interesting to examine the motion of light in *solid*, transparent, rapidly moving substances. In this connection experiments with rapidly moving quartz and glass have been made by Miss SNETHLAGE and myself. In this communication I will give the description of the apparatus with which these experiments have been made. It may be well to call attention to a few points referring to FIZEAU's experiment with water.

Let c be the velocity of light in vacuo, μ the index of refraction of the water, w the velocity of the water with respect to the tube in which it moves; then the velocity of propagation of the light with respect to the tube is according to FRESNEL:

$$\frac{c}{\mu} \pm \left(1 - \frac{1}{\mu^2}\right) w \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

In this the upper or the lower sign is to be taken according as the water and the light move in the same or in opposed directions.

In 1895 LORENTZ demonstrated that FRESNEL's convection coefficient in a dispersive medium must be replaced by:

$$1 - \frac{1}{\mu^2} - \frac{\lambda}{\mu} \frac{d\mu}{d\lambda}$$

This changes formula (1) into:

$$\frac{c}{\mu} \pm \left(1 - \frac{1}{\mu^2} - \frac{\lambda}{\mu} \frac{d\mu}{d\lambda}\right) w \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The experiments made by FIZEAU in 1851, plead in favour of

P. ZEEMAN: "The Propagation of Light in Moving Transparent Solid Substances. I. Apparatus for the Observation of the Fizeau-Effect in Solid Substances".

formula (1). MICHELSON and MORLEY's investigation of 1886, performed with MICHELSON's interferometer in one of the numerous forms into which as a real PROTEUS this wonderful instrument is capable of being changed, gave with white light a value of the convection coefficient which was in excellent agreement with the coefficient that follows for yellow sodium light from formula (1).

Experiments that have been carried out by me with different colours ranging from violet to red, and in which the axial velocity of water in the tube was directly measured have been communicated by me in different papers to this Academy [1]). The validity of the formula (2) with the term of dispersion could be demonstrated with an accuracy exceeding 0.5 %. The optical effect that is measured in these experiments, is a displacement of interference bands, which is given in parts of the distance of two bands by the formula:

$$\frac{4l}{\lambda \,.\, c}\left(1 - \frac{1}{\mu^2} - \frac{\lambda}{\mu}\frac{d\mu}{d\lambda}\right)\mu^2 w, \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

in which l represents the length of the whole liquid column which is in motion.

2. The apparatus that has been used for the investigation of the motion of light in solid substances, is shown diagrammatically in

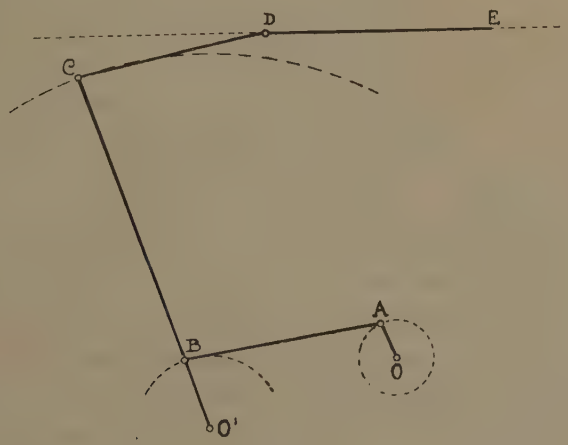

Fig. 1.

fig. 1 on a scale of $^1/_{10}$, and might also be used with simple modifications for the investigation of costly liquids and compressed gases.

The moving, transparent substance is rigidly connected with a piece *DE*, and can therefore rapidly move to and fro parallel to

[1]) ZEEMAN, These Proc. **17**, 445, 1914; **18**, 398, 1915; **18**, 711, 1915; **18**, 1240, 1916; **19**, 125, 1916.

the dotted line, while a beam of light traverses the substance parallel to DE.

The piece DE is moved to and fro, as it is coupled with the rods DC and $O'C$. Normal to the plane of the drawing, axes have been fixed at O and O' in a very strongly constructed frame, on which the bed is fastened, along which DE moves. The axis in O is rotated by a 3 H.P. motor, so that point A describes a circle; B, connected with A by the rod AB, acquires a movement backward and forward, which is transferred to C enlarged.

The piece DE of fig. 1 is shown diagrammatically in fig. 2 seen from above on a scale of $\frac{1}{10}$. In A and B there are bronze shoes which can slide along steel guides. All these have been constructed with great care, so that a rectilinear, horizontal motion of the shoes can be obtained. The rods of the transparent substance, which

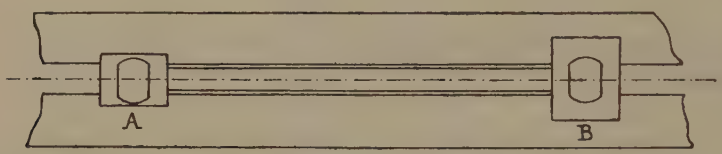

Fig. 2.

rest on a wooden block which is connected with 4 screws to A and B, participate in this motion.

The Plate, which is annexed to this communication, gives a general survey of the apparatus. The thickness of the guides amounts to 9 mm., the width to 70 mm., the length to 1.84 m. They rest on heavy rectangularly bent pieces, which constitute the sides of the bed, and which for greater firmness are connected by very solid pieces about half a meter long, which are bent twice rectangularly. These pieces are arranged on the lefthand at the bottom side, on the righthand on the upper side of the bed, as is to be seen in the Plate. The entire length of the upper part of the apparatus is 2.30 m., the length of the stroke is about 1 meter.

In order to ensure the regular movement of the apparatus it appeared to be necessary to provide it with two fly-wheels, a large one seen in the foreground, and a smaller one fastened on the other side of the axis of rotation, and just visible on the Plate. The whole apparatus it fastened with solid bolts to a granite slab, cemented to the large pillar of the laboratory.

As appears from a consideration of fig. 1, the rate of motion of the shoe is variable, with two, slightly differing maxima of velocity,

one in going, and one in returning along the guides. The maximum velocity is practically constant over a distance of about 20 cm. When the fly-wheel performs 184 revolutions a minute, the maximum velocity rises to somewhat more than 10 meters per second. This is the highest value that can be reached.

The driving apparatus was constructed by the works Werkspoor at Amsterdam. The execution of the mechanical parts had to be adapted to the optical requirements in the Laboratory.

With regard to the optical arrangement of the experiment we may refer to my former communications on the FIZEAU-effect. The quickly moving transparent, solid substance takes the place of the running water of the earlier experiments. The length of the moving column of quartz or glass ranged between 100 and 140 cm. in different experiments. After the successive application of numerous improvements it was possible to cause the beams of light to interfere through the moving quartz or glass, and to obtain pure interference lines at the greatest velocity which the apparatus admits.

The experiment comes to this, that the interference bands are photographed twice, first with a movement of the column to the right, and then with a movement to the left. These photos should be taken by admitting light during a time of the order of one hundredth of a second and at the moment of the maximum velocity.

The optical effect to be expected, when l represents the length of the moving transparent substance is:

$$\frac{4l}{\lambda . c}\left(\frac{1}{\mu}-\frac{1}{\mu^3}-\frac{\lambda}{\mu^2}\frac{d\mu}{d\lambda}\right)\mu^2 w \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

It appears from this formula, which will be proved later on (see II), that the optical effect is approximately proportional to $\mu-1$. In FIZEAU's experiment the optical effect is proportional to μ^2-1 according to (3). This difference is connected with the fact that in FIZEAU's experiment in its usual form, the velocity in a definite point of space is always the same, whereas in the experiment now considered the light must overtake the moving bar.

As regards the optical effect observed, the method considered now will accordingly be two and a half times less favourable for a value of $\mu = 1{,}5$ than FIZEAU's usual method, because $\frac{\mu^2-1}{\mu-1} = \mu + 1$. This is more or less compensated by an advantage with regard to the dispersion term. As follows from formulae (3) and (4), the ratio of the dispersion term to the principal term is in the second case 1,6 times that in the former experiment.

We shall now discuss a few more particulars of the arrangement and the use of the apparatus.

3. *Determination of velocity.* In order to get an insight into the course of the velocity in the movement along the guides the position of *DE* (see fig. 1) was determined corresponding with 16 different, equidistant positions of the fly-wheel. The graph indicating the connection between the positions of the fly-wheel and the deviations, has about the shape of a sinusoid, but the two halves of the curve are not symmetrical, and in particular, the course of the graph in the neighbourhood of the two boundary values is not exactly the same, as already appears from a consideration of fig. 1, when *A* is imagined to move along the dotted circle.

The velocity-time curve can be graphically derived from the path-time curve. At the maxima the velocity is practically constant over a distance of 20 cm., of which only 10 cm. are used. As was already stated the maximum velocity amounts to 1000 $\frac{\text{cm.}}{\text{sec.}}$ for 184 revolutions per minute, and proportionally the calculated velocity can be derived for another number of revolutions. Whether really the maximum velocity should be taken into account in the calculation, depends further also on the position of the moving column at the moment that the shutter, before the objective of the telescope, transmits light. In some cases this position did not correspond to that of the maximum velocity, which circumstance was of course taken into account in the interpretation of the photos.

In the most accurate experiments the maximum velocity of the column was directly measured (for the method used see one of the following communications), which renders us independent of the supposition that the fly-wheel possesses a constant angular velocity.

It appeared in the experiments that the machine ran more uniformly when the fly-wheel rotates clockwise (seen from the side of the larger fly-wheel) than in the opposite direction. Of course this favourable direction was always used.

4. *Shutter.* Only at the moment that the machine has its greatest velocity may the light be admitted to the photographic plate. The following arrangement was made for this purpose. The axis of the fly-wheel is provided with a toothed wheel, which engages with a second toothed wheel with double the number of teeth. An insulated brass ring with cams is fitted on each side of this second wheel. The cams on the two rings are placed

diametrically opposite one another so that they take each other's places as regards level after every whole revolution of the fly-wheel, and can make contact with a suitably fixed sliding contact.

By means of the cam an electric current is closed in a circuit containing also windings of a coil that acts electromagnetically on a shutter or light interrupter. When the second cam makes contact, the current passes through a second coil, which closes the shutter. Every time the fly-wheel, and consequently the moving column, have arrived at the *same position* and move *in the same direction*, the shutter is opened, and light is allowed to pass for a moment. The intensity of the interference figure not being strong enough to give a satisfactory photo with light that has been admitted once, a photo is taken e.g. thirty times successively on the same plate. As the light is let through three times a second, this takes only ten seconds.

To take the second photo, i.e. when the column of quartz moves in the opposite sense, a duplicate arrangement is used, placed symmetrically with respect to the one sketched. By means of a double-pole throw-switch it becomes possible to make one series of photos succeed the other immediately without loss of time and without stopping the machine. Only the photographic plate must be moved a little. The two large toothed wheels, and between them the small one can be distinguished on the Plate near the bearing at the bottom on the righthand side.

As some time passes between the moment that contact is effected by the cam and the opening of the shutter, this time must be taken into account. To do so it is necessary to perform a phase determination.

5. *Phase determination*, i.e. to ascertain by a separate experiment, if really at the moment of the greatest velocity the light passes through the shutter. For this purpose the shutter is put at the place of the greatest velocity by the side of the bed. The wooden beam containing the transparent substance, is provided with a black screen with an opening. A glowlamp is placed in the line: shutter-place of the greatest velocity, a line which is normal to the longitudinal direction of the apparatus. When the machine is running and when the shutter is in action, the lamp must be observed through it. When it does not work at the right moment, the field of view remains dark. Then the phase can be improved, and can at last be made accurate by gradually shifting the large toothed wheel with respect to the small one. This causes the contact to be formed at another moment.

6. *Observation and Photography of the Interference-bands.* The shutter is placed before the objective of the telescope, which was also used before to record the FIZEAU-effect [1]). In the focal plane of the telescope, which is provided with a negative achromatic system of lenses to increase the effective focal distance, a system of wires has been placed, which is photographed at the same time with the interference bands.

The position of the interference fringes with respect to the wires is determined. Immediately behind the wires the photographic plate is in a plateholder mounted quite independently of the telescope with the cross wires. The photographic plate can be put in the required position without the telescope being touched, and be shifted to take the successive photos.

The telescope, the plate-holder, the interferometer, and the glass rectangular prism, in which the interfering beams are reversed, are mounted on separate freestone piers, which are cemented to the large pier. This last mentioned prism, which also served in the earlier experiments, is visible on the righthand side of the Plate.

7. *Measurement of the time that the shutter or interrupter is opened.* This time, which is of the order of 1 or 2 hundredths of a second, is dependent on the current in the coils of the circuit of the interrupter and can be regulated by this and by the change of the width of the opening in the moving screen of the interrupter. For the determination of the time the interrupter is placed before the lens of a camera, with which a small lamp is photographed, which revolves on a disk with known velocity. During the time that the interrupter is opened, the lamp describes part of a circle, the length of which is measured.

8. *Checking and regulating of the apparatus.* After the interference fringes have been made as distinct as possible, the beams passing only through the air, a compensator is placed in one beam, consisting of a plane parallel glass plate 7 m.m. thick and with a diameter of 25 m.m., made by Hilger. This plate can be rotated round a vertical and horizontal axis, and enables us to change the slope and the distance of the interference fringes in a simple way. In many cases it was unnecessary to insert this compensator, as the desired interference fringes were already obtained with the interferometer alone. Then the column of quartz and glass

[1]) Cf. ZEEMAN. These Proc. **18**, 400, 1915.

is introduced into the beam of light, precautions being taken which will be mentioned in communication II. It must then first of all be ascertained whether the apparatus and particularly the guides satisfy the high requirements on which the efficacy of the whole arrangement depends.

With a slow movement of the shoes with the quartz column along the guides the interference fringes did not remain stationary, but changed with regard to distance and slope. When the apparatus had first been put together, this movement of the bands was very great. It is clear that it must also occur for a perfectly homogeneous column bounded by parallel planes, when the movement does not take place along a perfectly straight line. For then the whole column of a length of more than 100 cm. acts as a compensator of exceedingly great thickness. In order to make the circumstances as favourable as possible the steel plates of the guides were laid on the supporting plates about in the correct position. The bolts which were used to keep the apparatus in place, are then screwed down till a sensitive level, which could be placed in longitudinal and in transverse direction, indicated a plane as much as possible horizontal, determined by the upper planes of the steel guides. It was then examined if the inner edges of the guides were as perfectly straight and parallel as possible, and improvements were made in this respect by filing and grinding. At last the free play of the bronze shoes in their movement along the guides was removed as much as possible. Great improvements were successively made to the apparatus in this direction, so that rotation and change of distance of the fringes became comparatively slight. It was, however, impossible to have the interference fringes quite steady when the apparatus was slowly moved. This is, however, not necessary; what is required after all is that the same positions of the interference bands are found again when the shoes have returned to the same point of the guides. The results prove that this is actually the case, and that the occasional deviations fall now in one sense, now in another.

The excellent definition of the interference fringes, recorded with the quickly moving apparatus in itself proves already, that every time about the same position of the bands is obtained, as 20 or 30 images are superposed (see above § 4), which could never produce a definite image, when the single light impressions were not almost identical. Sometimes the system of fringes proved to be rotated, and then the photo had to be rejected. Of course care had also to be taken that the guides were well oiled, and there is one more *dynamic particularity* that had to be seen to. When the motor has

been started for the first time, then the apparatus hardly ever gains the maximum velocity, which corresponds to 184 revolutions of the fly-wheel per minute. It gives the impression that the apparatus is hampered by a resistance, e.g. only 140 revolutions are made. The starting is then repeated a few times, and at the third or fourth attempt the machine suddenly runs very smoothly without jerking. Then the feeling of uneasiness of the operator at the exceedingly rapid motion of the large apparatus so close to the delicate optical parts of the interferometer, has abated somewhat and the experiment can begin. [1])

[1]) The experiment is not entirely without danger. When the experiment with glass was to begin, four beautifully finished glass cylinders 20 cm. long and 2.5 cm. thick were placed in the wooden shoe, and optically adjusted. In the very first experiment with this glass column one of the glass cylinders, which evidently had not been properly fastened, got loose, while the apparatus moved at full speed; it smashed all the other pieces and knocked the brass end-pieces off the shoe. The glass cylinders were entirely smashed, the work of months was destroyed. It was a wonder that the interferometer and the glass rectangular prisms remained undamaged.

Mathematics. — "*Ueber die Struktur der perfekten Punktmengen*" (dritte Mitteilung [1]). By Prof. L. E. J. BROUWER.

(Communicated in the meeting of October 25, 1919).

§ 1.

Ein *Flächensystem* bzw. eine *Fläche* ist im folgenden definiert mittels eines solchen zweidimensionalen Fragmentes [2]) bzw. zusammenhängenden zweidimensionalen Fragmentes, in welchem die in einem willkürlich gewählten Elementeckpunkte mündenden Elementseiten entweder alle oder alle bis auf zwei je zwei Elementen gemeinsam sind. Im ersteren Falle sprechen wir von einem *gewöhnlichen*, im letzteren von einem *aussergewöhnlichen* Elementeckpunkt, während wir eine Elementseite gewöhnlich oder aussergewöhnlich nennen, je nachdem sie zu zwei oder zu einem einzigen Elemente gehört. Das Flächensystem bzw., die Fläche besteht aus dem Fragmente *mit Ausnahme* der aussergewöhnlichen Elementseiten und Elementeckpunkte, welche zusammen die *Grenze* des Flächensystems bzw. der Fläche bilden.

Ein Flächensystem bzw. eine Fläche, deren Elemente Grundsimplexe einer simplizialen Zerlegung des Flächensystems oder der Fläche ω sind, wird ein *Teilflächensystem* bzw. eine *Teilfläche* von ω genannt werden.

Sei α ein auf der Fläche ω nicht-negativer Charakteristik gelegenes und auf ω kein elementares [3]) Restgebiet bestimmendes, abgeschlossenes Kontinuum, $\alpha' = \omega, \alpha'', \alpha''', \ldots$ eine α approximierende Folge von Teilflächen von ω, durch welche also α als $\mathfrak{D}(\alpha', \alpha'', \ldots)$ bestimmt ist, so dürfen wir für jedes ν annehmen, dass $\alpha^{(\nu)}$ auf ω kein elementares Restgebiet bestimmt, dass $\alpha^{(\nu+1)}$ eine Teilfläche von $\alpha^{(\nu)}$ ist und dass die Grenzen von $\alpha^{(\nu)}$ und $\alpha^{(\nu+1)}$ keinen Punkt gemeinsam haben. Sei $k^{(\nu)}$ die Charakteristik von $\alpha^{(\nu)}$, $m^{(\nu)}$ die maximale Anzahl von einander nicht treffenden, zusammen nicht zerstückelnden einfachen geschlossenen Kurven von $\alpha^{(\nu)}$, so kann für wachsendes ν

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[1]) Für die erste und zweite Mitteilung vgl. diese Proceedings XII, S. 785; XIV, S. 137.

[2]) Vgl. Math. Annalen 71, S. 306.

[3]) Ein Teilgebiet von ω heisst elementar, wenn es nur auf ω zusammenziehbare einfache geschlossene Kurven enthält.

weder $k^{(\nu)}$ noch $m^{(\nu)}$ zunehmen, so dass eine endliche positive Zahl g existiert mit der Eigenschaft, dass $k^{(g+\mu)} = k^{(g)}$ und $m^{(g+\mu)} = m^{(g)}$ für jedes nicht negative μ. Hieraus folgt, dass die von $\alpha^{(g+\mu+1)}$ in $\alpha^{(g+\mu)}$ bestimmten Restflächen alle Zylinderflächen sind, dass die topologische Gestalt von $\alpha^{(g+\mu)}$ für jedes μ gleich derjenigen von $\alpha^{(g)}$ ist und dass $\alpha^{(g+\mu+1)}$ aus $\alpha^{(g+\mu)}$ durch Zurücknahme der Ränder nach Innen entsteht, so dass zwischen den $\alpha^{(g+\nu)}$ und α die folgenden Beziehungen existieren:

1. Zu jeder Kombination $(\varepsilon, \varepsilon_1, \mu)$ gibt es ein solches nur von ε und μ abhängendes und für festes μ mit ε gegen 0 konvergierendes ε_2, dass jede ε-Kette [1]) von $\alpha^{(g+\mu)}$ mittels einer endlichen Folge von ε_2-Abänderungen [1]) innerhalb $\alpha^{(g+\mu)}$ in eine ε_1-Kette von α übergeführt werden kann.

2. Zu jedem ε gibt es ein solches mit ε gegen 0 konvergierendes ε_3, dass jede ε-Kette eines $\alpha^{(g+\mu)}$ mittels einer endlichen Folge von ε_3-Abänderungen innerhalb $\alpha^{(g+\mu)}$ in eine Kette von α übergeführt werden kann.

3. Jedes $\alpha^{(g+\mu)}$ besitzt die gleiche minimale Multiplizität der Basis der Zyklosis [2]), wie α.

4. Zu jeder Kombination eines hinreichend kleinen ε', eines ε'_1 und eines μ gibt es ein solches nur von ε' und μ abhängendes und für festes μ mit ε' gegen 0 konvergierendes ε_2, dass jedes System von $(\varepsilon, \varepsilon')$-Fundamentalketten [2]) von $\alpha^{(g+\mu)}$ mittels einer endlichen Folge von ε_2-Abänderungen innerhalb $\alpha^{(g+\mu)}$ in ein System von $(\varepsilon_1, \varepsilon'_1)$-Fundamentalketten von α übergeführt werden kann.

5. Zu jedem ε' gibt es ein solches mit ε' gegen 0 konvergierendes ε'', dass jedes System von $(\varepsilon, \varepsilon')$-Fundamentalketten eines $\alpha^{(g+\mu)}$ mittels einer endlichen Folge von ε''-Abänderungen innerhalb $\alpha^{(g+\mu)}$ in ein System von $(\varepsilon, \varepsilon'')$-Fundamentalketten von α übergeführt werden kann.

6. Zu jedem ε gibt es ein solches ε^0, dass jedes System von $(\varepsilon, \varepsilon')$-Fundamentalketten von α gleichzeitig ein System von $(\varepsilon^0, \varepsilon')$-Fundamentalketten eines jeden $\alpha^{(g+\mu)}$ ist.

Wir bringen diese Beziehungen zum Ausdruck, indem wir α als *zyklomatisches Extrakt* von $\alpha^{(g)}$ bezeichnen.

Sei nunmehr α ein willkürliches auf ω gelegenes abgeschlossenes Kontinuum, α_1 die Vereinigung von α und den elementaren Restgebieten von α, $\alpha_1^{(g_1)}$ eine α_1 als zyklomatisches Extrakt enthaltende Teilfläche von ω. Alsdann gibt es zu jeder Kombination $(\varepsilon, \varepsilon_1)$ ein

[1]) Vgl. Math. Annalen 72, S. 422.

[2]) Ibid., S. 424.

solches mit ε gegen 0 konvergierendes ε_2, dass jede ε-Kette von $\alpha_1^{(g_1)}$ mittels einer endlichen Folge von ε_2-Abänderungen innerhalb ω in eine ε_1-Kette von α übergeführt werden kann. *Wir werden auch α ein zyklomatisches Extrakt von $\alpha_1^{(g_1)}$ nennen.*

Insbesondere werden wir ein zyklomatisches Extrakt eines Elementes als *Elementarkontinuum*, ein zyklomatisches Extrakt einer Zylinderfläche als *Zylinderkontinuum* bezeichnen.

§ 2.

Sei α eine auf der Fläche ω nicht-negativer Charakteristik gelegene, auf ω kein elementares Restgebiet bestimmende, abgeschlossene Punktmenge, $\alpha' = \omega, \alpha'', \alpha''', \ldots$ eine α approximierende Folge von Teilflächensystemen von ω, so dürfen wir für jedes ν annehmen, dass $\alpha^{(\nu)}$ auf ω kein elementares Restgebiet bestimmt, dass $\alpha^{(\nu+1)}$ ein Teilflächensystem von $\alpha^{(\nu)}$ ist und dass die Grenzen von $\alpha^{(\nu)}$ und $\alpha^{(\nu+1)}$ keinen Punkt gemeinsam haben. Sei $\beta^{(\nu)}$ das aus $\alpha^{(\nu)}$ durch Tilgung derjenigen Stücke, welche die topologische Gestalt eines Elementes oder eines Zylinders besitzen, hervorgehende Flächensystem, S ein willkürliches Stück von $\beta^{(\nu)}$, $k(S)$ die Charakteristik von S, $m(S)$ die maximale Anzahl von einander nicht treffenden, zusammen nicht zerstückelnden einfachen geschlossenen Kurven von S, so sind die Zahlen $k(S)$ und $m(S)$ beide nicht-negativ und nicht beide gleich Null. Mithin existieren zwei ganze nicht-negative Zahlen g und h mit der Eigenschaft, dass jedes $\beta^{(g+\mu)}$ sich aus h Stücken $\beta_1^{(g+\mu)}, \beta_2^{(g+\mu)}, \ldots . \beta_h^{(g+\mu)}$ zusammensetzt, dass die topologische Gestalt von $\beta_s^{(g+\mu)}$ für jedes μ und jedes s gleich derjenigen von $\beta_s^{(g)}$ ist, dass $\beta_s^{(g+\mu+1)}$ aus $\beta_s^{(g+\mu)}$ durch Zurücknahme der Ränder nach Innen entsteht und dass das als $\mathfrak{D}(\beta_s^{(g)}, \beta_s^{(g+1)}, \ldots)$ bestimmte Stück von α ein zyklomatisches Extrakt von $\beta_s^{(g)}$ ist.

Das hiermit erhaltene Resultat kann in der folgenden Form, welche, wie eine triviale Ueberlegung zeigt, für eine Fläche ω negativer Charakteristik ihre Gültigkeit behält, ausgedrückt werden:

Zu jeder auf einer Fläche ω gelegenen abgeschlossenen Punktmenge α existiert ein Teilflächensystem ψ von ω mit der Eigenschaft, dass α sich aus erstens von den Stücken von ψ je einem zyklomatischen Extrakt, zweitens einer Menge von Elementarkontinuen und Zylinderkontinuen zusammensetzt.

Ist insbesondere ω eine Kugel, so sind alle Stücke von α Elementarkontinua.

Ist ω eine projektive Ebene, so sind entweder alle Stücke von α

Elementarkontinua, oder alle bis auf eines, das ein zyklomatisches Extrakt von ω ist.

Ist ω ein Torus, so sind entweder alle Stücke von α Elementarkontinua, oder alle bis auf eines, das ein zyklomatisches Ektrakt von ω' ist, oder endlich alle bis auf eine abgeschlossene zyklisch geordnete Menge von Zylinderkontinuen; die Teilzylinder von ω, von denen diese Zylinderkontinua zyklomatische Extrakte sind, sind alle auf ω stetig ineinander überführbar.

Mathematics. — "*Ueber Transformationen ebener Bereiche*". By B. VON KERÉKJÁRTÓ. (Communicated by Prof. L. E. J. BROUWER).

(Communicated in the meeting of October 25, 1919).

In der vorliegenden Note wird eine Anwendung [1]) gemacht vom folgenden BROUWERschen Fixpunktsatze:

Eine eineindeutige stetige Abbildung der abgeschlossenen Kreisscheibe auf einen Teilbereich derselben lässt wenigstens einen Punkt invariant.

Mit Hilfe dieses Satzes beweisen wir nämlich das folgende

THEOREM. *Eine eineindeutige stetige Abbildung eines von endlichvielen Jordanschen Kurven begrenzten abgeschlossenen ebenen Bereiches auf einen Teilbereich desselben, bei welcher die Grenzkurven des ursprünglichen und des Bildbereiches paarweise äquivalent sind, jedoch eine und nur eine Grenzkurve in eine äquivalente übergeht, lässt wenigstens einen Punkt invariant.*

(Hierbei sollen zwei einander nicht kreuzende Kurven *äquivalent* genannt werden, wenn in ihrem Zwischengebiete keine Grenzkurve liegt).

C_1 sei die äussere Grenzkurve des gegebenen Bereiches, ihr Bild C_1' sei mit ihr äquivalent; die übrigen Grenzkurven seien $C_2, C_3, \ldots C_n$, ihre Bilder $C_2', C_3', \ldots C_n'$. Man erweitere die gegebene Abbildung durch eine an sie anschliessende Abbildung des Innern von C_α auf das Innere von C'_α ($\alpha = 2, 3, \ldots n$). Hiermit erhält man eine eineindeutige stetige Abbildung des Innern von C_1 auf das Innere von C_1', welche nach dem obigen BROUWERschen Satze wenigstens einen Punkt invariant lässt; dieser Fixpunkt kann aber nicht im Innern von C_α ($\alpha \neq 1$) liegen, gehört somit zum ursprünglichen abgeschlossenen Bereich.

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[1]) Für eine analoge Anwendung nebst daraus gezogenen Konsequenzen vgl. Math. Annalen 80, S. 34.